ALGEBRA BY VISUAL AIDS

Book Two

VISUAL AID SERIES

Edited by LANCELOT HOGBEN

ALGEBRA BY VISUAL AIDS

Book 1: The Polynomials

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ALGEBRA BY VISUAL AIDS

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Book Two

THE CONTINUUM

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Contents

BOOK II

THE CONTINUUM

CHAPTER 8

From Steps to Slopes

§ 1. REMINDERS

In Part I of *Algebra by Visual Aids* we dealt with various families of numbers. The build-up of each family had a definite rule, e.g. the series

$$1 \qquad 8 \qquad 27 \qquad 64 \qquad 125$$

is formed by multiplying each of the numbers 1, 2, 3, 4, 5 by itself twice over. We can show each number by a pattern of dots, the above as a series of cubic patterns, displayed in Chart 5 (Book I). We can also specify the series by means of an algebraic formula:

$$(\mathrm{Cu})_n = n^3$$

All our series of Book I are expressible by means of:

- (*a*) a set of numbers;
- (*b*) a verbal rule;
- (*c*) a set of dot patterns;
- (*d*) an algebraic *formula* or *equation*.

Of these four, the last is by far the shortest and most compact. Many people are completely mystified by Algebra because nobody has troubled to show them the meaning of the formulae. The charts in Book I should have given you enough confidence to go straight ahead with all sorts of formulae. So you should now be able to handle algebraic expressions without having to rely on more cumbersome ways of expressing the same thing.

For our present purpose we can regard the formulae of Book I as expressing series of *numbers*. Each number in the series is obtainable from the formula by giving a definite numerical value to the rank n, e.g. in the formula:

$$(\mathrm{Cu})_n = n^3$$

We can give n the value 4. We then have:

$$\begin{aligned}(\mathrm{Cu})_4 &= 4^3\\ &= 4 \times 4 \times 4 = 64\end{aligned}$$

This process is known as *substitution*. By substitution we can obtain the number series from the formula, i.e. we put $n = 1, 2, 3 \ldots$ In this way we can think of the formula as expressing a *law of growth*. The patterns bring this out. Each of them grows from the previous one by adding a layer of dots. Now this process of *growth* is important and is very well worth studying. All living things grow and so do many non-living things. A building grows, brick by brick. A city grows, street by street. A crystal grows, layer by layer. The day grows, hour by hour. A baby grows, pound by pound. In the first book our diagrams expressed *number-shapes*. The diagrams of Book II express GROWTH-SHAPES.

Ex. 8.11. Substitution Practice

Find the values of the following expressions by substituting the given values of n.

1. $4 + 3n$ values $n = 3, 4, 5, 6.$
2. $10 + 5n$ values $n = 2, 4, 6, 8.$
3. $6 - 3n$ values $n = 0, 3, -3, 6.$
4. $-12 + 4n$ values $n = -5, 0, 5, 10.$
5. $-9 - 6n$ values $n = -20, -10, 0, 10.$
6. $\frac{n}{2}(n + 1)$ values $n = 0, 1, 2, 3.$
7. n^2 values $n = 0, 2, 4, 6.$
8. $(n + 2)(n - 2)$ values $n = -8, -4, 0, 4.$
9. $n(n + 1)(n + 2)$ values $n = -1, 0, 1, 2.$
10. n^4 values $n = -3, -1, 1, 3.$

§ 2. GROWTH DIAGRAMS

The simplest kind of growth is *growth by equal steps*. This is the kind we are concerned with in the first two chapters of Book II. In Chapter 6, we studied series of the type called Arithmetical Progressions, e.g.:

$$3, \quad 5, \quad 7, \quad 9, \quad 11$$

You get each number of this series by adding 2 to the previous one. We called the number added the *common difference*. If the series starts with A_0 and adds d at each step our formula for the nth term after the initial one (A_0) is:

$$A_n = A_0 + nd$$

For example, if $A_0 = 8$ and $d = 3$, the series is:

$$(8), \quad (8 + 3), \quad (8 + 2 \cdot 3), \quad (8 + 3 \cdot 3), \ldots$$
$$8 \quad 11 \quad 14 \quad 17 \ldots$$

We can also continue the series *backwards* by *subtraction*, and watch it shrinking instead of growing:

$$A_{-1} = A_0 - d \qquad = 8 - 3 = 5$$
$$A_{-2} = A_0 - 2 \cdot d \qquad = 8 - 6 = 2 \qquad \text{etc.}$$

In Book II, we shall represent *whole* numbers by a new shape, viz.: a vertical strip or column. The length of the column will represent the numerical value of any term in a series. We can use any unit of length we like, e.g. 1 *inch*, 1 *cm.*, or some smaller unit, e.g. 0·1 *in.* The number of vertical units assigned to the column stands for the numerical value. A *series* of numbers can thus be represented by an array of columns placed side by side. The position of each column gives its rank, the first (on the left) being of rank $n = 0$, the next $n = 1$, the next $n = 2$, and so on. Charts 44 and 45 show the growth diagrams, so constructed, for the Arithmetical Progressions:

$$A_n = 8 + 3n; \quad A_n = 32 + \tfrac{9}{8}n$$

Note the following points:

1. We can choose any scale we like. If the scale is small we can represent more terms on the same piece of paper.
2. We can put the starting-point A_0 anywhere we like. The further to the left we put it, the more terms we can put on the right, and vice versa.
3. The *contour* of the growth diagram rises up by *regular steps*, expressing the fact that the Arithmetical Progression increases by *equal differences*.
4. If the common difference were more than 3, the steps would go up more steeply; if less than 3 they would go up less steeply *on the same scale*.
5. We call a diagram of this type a *column-graph* or *histogram* (Greek *histos* = a mast and *gramma* = a letter or message).

Ex. 8.21. Histograms of Number-Series

Take a quarter-inch as unit and draw histograms for the following series:

$n =$	0	1	2	3	4	5
1. $A_n =$	0	2	4	6	8	10

2. $A_n =$	5	9	13	17	21	25
3. $A_n =$	3	6	9	12	15	18
4. $A_n =$	8	12	16	20	24	28
5. $A_n =$	1	4	7	10	13	16
6. $A_n =$	6	7	8	9	10	11
7. $A_n =$	0	5	10	15	20	25
8. $A_n =$	17	15	13	11	9	7
9. $A_n =$	6	6	6	6	6	6
10. $A_n =$	30	24	18	12	6	0

From the last exercise you will realize that if you keep to the same unit some histograms take up a great deal more space than others. This is inconvenient. Therefore it is best to adapt your unit to the value of the numbers. With large numbers use a small unit, e.g. $\frac{1}{10}$ in., and vice versa. For the present, use the same unit for n (*horizontal* direction) as for A_n (*vertical*).

You can draw histograms for negative as well as for positive numbers. Chart 46 gives you the histograms for the 4 series in the table below:

	Values						*Formulae*
1.	A_0 -20	A_1 -12	A_2 -4	A_3 $+4$	A_4 $+12$	A_5 $+20$	$A_n = -20 + 8n$
2.	A_{-3} -1	A_{-2} $-\frac{2}{3}$	A_{-1} $-\frac{1}{3}$	A_0 0	A_1 $\frac{1}{3}$	A_2 $\frac{2}{3}$	$A_n = 0 + \frac{1}{3}n$
3.	A_2 12	A_3 9	A_4 6	A_5 3	A_6 0	A_7 -3	$A_n = 18 - 3n$
4.	A_{-1} 3	A_0 1	A_1 -1	A_2 -3	A_3 -5	A_4 -7	$A_n = 1 - 2n$

These histograms show you:

(*a*) The effect of scale;
(*b*) The plotting of negative terms;
(*c*) The shape of *shrinking* as well as *growing* series;
(*d*) Plotting from different starting-ranks.

Ex. 8.22. Histograms from Formulae

Find, by substitution, the values of the following formulae for the given values of n. Then plot the histogram on the scale given. Label each column as in Chart 45.

1. $A_n = 3 + 2n$.	Values: $n = 0$ to 5.	Scale: $\frac{1}{4}$ inch = 1 unit.
2. $A_n = -5 + 3n$.	Values: $n = 2$ to 7.	Scale: $\frac{1}{10}$ inch = 1 unit.
3. $A_n = 10 - 2n$.	Values: $n = -1$ to 4.	Scale: $\frac{1}{4}$ inch = 1 unit.
4. $A_n = -40 + 10n$.	Values: $n = 2$ to 5.	Scale: $\frac{1}{10}$ inch = 1 unit.
5. $A_n = 5 - 1n$.	Values: $n = 1$ to 8.	Scale: $\frac{1}{2}$ inch = 1 unit.

§ 3. SAVINGS ACCOUNTS

So far our histograms have differed in only three ways:

1. *The width of the strips.* This depends simply on our choice of scale and is thus not a real difference between the histograms themselves.
2. *The starting-rank.* This depends on the value of A_0. It does not affect the shape or size of the histogram but merely its position.
3. *The steepness of the steps.* This depends on the amount we add at each step, i.e. the common difference d. Also, if d is negative the steps go down instead of up.

To give a worldly meaning to these diagrams we can take the value A_0 to represent the amount of money a man has in the bank on January 1st. If it is negative it is an overdraft, i.e. an amount which he *owes* to the bank. Then suppose he saves out of his wages a fixed sum every month, e.g. £3, and pays this into the bank. This is represented by the step up from one column to the next. The height of each column, which stands for the total amount in the bank, corresponds to the height of a pile of coins. If the depositor starts with £8 on January 1st ($A_0 = 8$) and adds £3 each month his bank balance goes up like this:

Month	0	1	2	3	4
Balance (£)	8	11	14	17	20

Suppose someone asks: what was the balance on November 1st, i.e. 2 months earlier than January 1st? The answer would be $8 - 2(3) = 8 - 6 = 2$. *Answer* £2.

All this is in Chart 44. We can now give a new meaning to each feature of this histogram:

1. The amount on January 1st (month 0) is A_0.
2. The *rate of saving* is d.
3. The number of months up to any particular date is n.

Notice particularly the rate of saving, d. The bigger this is, the more steeply the steps go up. Thus d represents the *increase* of the balance *per month*, or more generally:

$$d = \text{increase of } A_n \text{ for a unit increase of } n.$$

Notice, too, that d does *not* depend on A_0. Next notice what happens when we subtract the balance of one month from the balance of any other month, and divide the answer by the number of months in the interval, e.g. for the months April and June:

$$A_5 - A_3 = 23 - 17 = 6$$

No. of months: $5 - 3 = 2$.

(*Amount saved*) $\div$ (*time*) $= 6 \div 2 = 3$.

Take another pair of values, e.g. months 1 and 4 (February and May):

$$A_4 - A_1 = 20 - 11 = 9$$

$$4 - 1 \quad = 3$$

(*Amount saved*) $\div$ (*time*) $= 9 \div 3 = 3$.

You can easily verify that the answer is always the same. If you recall Chapter 7 in Part I you will know that numbers formed by dividing one number by another have a name. We call them *ratios*. Thus we can express the above rule by saying that *the ratio of Amount saved to time-interval is always the same*. It is, in fact, equal to the constant difference d.

When we have two changing quantities which have a fixed ratio we say that they *vary in direct proportion*, or that one is directly proportional to the other. This is sometimes represented by a special symbol $\propto$. Thus *Amount saved* $\propto$ *time-interval*, i.e.:

$$(A_n - A_m) \propto (n - m)$$

We can also express the same thing by saying that *the ratio is constant*. If C

stands always for the same number in this formula, we then call it a *constant*, and we then write:

$$\frac{(A_n - A_m)}{n - m} = C$$

We can sum up by saying that *the equal-step histogram is a growth-diagram of direct proportion or constant ratio.*

EX. 8.31. BANK BALANCE HISTOGRAMS (CHART 47)

Taking January 1, 1944, as $n = 0$ (where n = number of months) draw histograms for the following accounts. Monthly savings = d.

1. Balance on January 1, 1944 = £10. Monthly savings, d = £2.
   Plot from January to August.
2. Balance on April 1, 1944 = £15. Monthly savings, d = £5.
   Plot from April to December
3. Balance on October 1, 1943 = £7. Monthly savings, d = £4.
   Plot from October 1943 to October 1944.
4. Balance on December 1, 1943 = − £20. Monthly savings, d = £3.
   Plot from December 1943 to July 1944.
5. Balance on June 1, 1943 = £8. Monthly *withdrawal*, d = − £2.
   Plot from June 1943 to June 1944.

EX. 8.32. READING THE HISTOGRAMS

The following questions refer to the histograms which you have drawn in Ex. 8.31.

1. In histogram of No. 1 read off the balances represented by A_3, A_5, A_7. Find the amount saved from February 1st to June 1st.
2. In No. 2 read off A_4, A_6, A_8, and find savings from May 1st to October 1st.
3. In No. 3 read off A_{-2}, A_0, A_2, and find savings from November 1, 1943, to May 1, 1944.
4. In No. 4 read off A_{-1}, A_3, A_7, and find savings from February 1st to June 1st.
5. In No. 5 read off A_{-5}, A_0, A_5, and find the amount withdrawn for the whole year.

§ 4. CHANGE OF SCALE

For all our histograms so far we have used the *same scale* for A_n as for n. This is not always convenient. For example, if values of A_n are very large compared with values of n we cannot plot more than a few values without shooting off the page. Now there is no reason against making the A-scale different from the n-scale and in future we shall do so wherever necessary; and we can do so safely if we bear in mind one thing. *A different scale alters the shape of the growth-diagram.* From this you can see:

1. That if both scales are changed in the same ratio there is no change of shape but simply of size.
2. That if the A-scale is larger than the n-scale the diagram lengthens upwards.
3. That if the n-scale is larger than the A-scale the diagram lengthens sideways.

Thus we have to be particularly careful about interpreting the slope of *Direct-Proportion* Histograms. It is exaggerated if the A-scale is greater than the n-scale and reduced if the n-scale is greater than the A-scale. The eye is not then a safe guide and the ratio must always be found by dividing the number of A-units by the *corresponding interval*, i.e. by the number of n-units.

Ex. 8.41. Effect of Change of Scale

Draw the following histograms each on 3 different scales as given.

1.

$n =$	-2	-1	0	1	2
$A_n =$	-4	-2	0	2	4

Scale 1.	1 n-unit $= \frac{1}{2}$ in.	1 A-unit $= \frac{1}{2}$ in.
Scale 2.	1 n-unit $= \frac{1}{2}$ in.	1 A-unit $= 1$ in.
Scale 3.	1 n-unit $= \frac{1}{2}$ in.	1 A-unit $= \frac{1}{4}$ in.

2.

$n =$	-3	-2	-1	0	1	2	3
$A =$	-12	-8	-4	0	4	8	12

Scale 1.	1 n-unit $= \frac{1}{4}$ in.	1 A-unit $= \frac{1}{4}$ in.
Scale 2.	1 n-unit $= \frac{1}{2}$ in.	1 A-unit $= \frac{1}{4}$ in.
Scale 3.	1 n-unit $= \frac{1}{10}$ in.	1 A-unit $= \frac{1}{4}$ in.

3.

$n =$	-4	-3	-2	-1	0	1	2	3	4
$A =$	4	3	2	1	0	-1	-2	-3	-4

Scale 1.	1 n-unit $= \frac{1}{2}$ in.	1 A-unit $= \frac{1}{2}$ in.
Scale 2.	1 n-unit $= \frac{1}{4}$ in.	1 A-unit $= \frac{1}{4}$ in.
Scale 3.	1 n-unit $= \frac{3}{4}$ in.	1 A-unit $= \frac{1}{12}$ in.

4. Plot the histograms for the series $A_n = 40 + 4n$ for values of n from -15 to $+5$.

Scale 1.	1 n-unit $= \frac{2}{10}$ in.	1 A-unit $= \frac{1}{10}$ in.
Scale 2.	1 n-unit $= \frac{1}{10}$ in.	2 A-units $= \frac{1}{10}$ in.
Scale 3.	1 n-unit $= \frac{3}{10}$ in.	4 A-units $= \frac{1}{10}$ in.

5. Plot the histogram for the series $A_n = 2 - \frac{1}{10}n$ for the following values of n : -30, -20, -10, 0, 10, 20, 30.

Scale 1.	2 n-units $= \frac{1}{10}$ in.	1 A-unit $=$ 1 in.
Scale 2.	1 n-unit $= \frac{1}{10}$ in.	1 A-unit $= \frac{1}{2}$ in.
Scale 3.	4 n-units $= \frac{1}{10}$ in.	1 A-unit $= \frac{1}{4}$ in.

§ 5. FINER DIVISIONS LEADING TO GRAPHS

Till now our histograms have been based on *whole* number series; but we can equally well have fractions. Once you have gained skill in adjusting your units you will have no difficulty in plotting these. For example, suppose you invest £1 at 5 per cent simple interest. This means that you start with £1 and add 1s. to it each year, i.e. £$\frac{1}{20}$ or £$\frac{5}{100}$. If $n =$ no. of years, $A_n =$ amount after n years and $A_0 =$ £1, we have the series shown in Chart 47(*a*):

$$A_n = 1 + \tfrac{1}{20}n$$

$$\begin{array}{lcccccc} n = & 0 & 1 & 2 & 3 & 4 & 5 \\ A_n = & 1 & 1\tfrac{1}{20} & 1\tfrac{2}{20} & 1\tfrac{3}{20} & 1\tfrac{4}{20} & 1\tfrac{5}{20} \end{array}$$

We may want to invest our money for longer than 5 years. Chart 47(*b*) shows the histogram for 20 years. These numerous, fine divisions are very tedious to draw and if we wanted to go on to 100 years, it would be almost impossible. One way out of this difficulty is to group the years in fives as in (*c*). This is quite simple to draw; but it does not enable us to read off the exact amount for any *single* year. It shows us the values only for 5, 10, 15, 20 years. There is thus a limit to the usefulness of the histogram method. For large numbers involving fine divisions we need a new one. From (*d*) in Chart 47 you see what happens if the number and fineness of the steps in (*b*) increase sufficiently. The step contour of the histogram becomes blurred, merging into a *straight line*. We call this straight line *the Limiting Boundary of the Histogram*.

Such a *straight line* chart has a number of distinct advantages:

1. We can draw it simply by plotting two points and joining them.
2. It is easily adaptable to any scale and therefore covers any range of values.

3. It gives the correct value at *every* point.
4. We can use it for a large number of useful calculations based on the formula

$$A_n = A_0 + d \cdot n$$

Limiting lines like this are known as *graphs*. Chart 47(*d*) shows you how to change from the histogram to the graph. Chart 48 shows the graph corresponding to Chart 44.

Squared paper is a great convenience for graph-drawing. It is not absolutely necessary but it saves a great deal of time. Notice the following:

1. We draw a *y-axis* through the 0-value on an *x-axis*. This divides the paper into two parts:

   right-hand side for + values of x,
   left-hand side for − values of x.

2. In order to emphasize the difference between the step-by-step growth of the histogram and the smooth, continuous growth of the graph, we adopt a new letter x in place of n. The symbol x stands for any number which can be divided into an indefinite number of parts. In place of A_n we adopt the letter y. This also is indefinitely divisible.
3. Our formula $A_n = 8 + 3n$ now becomes:

$$y = 8 + 3x$$

4. The point at which the graph cuts the y-axis depends on the value of A_0. In Chart 48, $A_0 = 8$ and the graph cuts the y-axis at the point $y = 8$, $x = 0$.
5. The slope of the graph is the same as that of the histogram. On the same scale, it depends solely on d, the common difference. Thus the value of y in Chart 48 goes up 3 units for each *unit* increase in x.

If we use y_0 for the value of y when $x = 0$, y_0 is numerically the same as A_0 in our equal-step histogram, and the coefficient of x in the equation of the straight-line graph is the common difference d. We can therefore write it as:

$$y = y_0 + d \cdot x$$

Hence we can put:

$$y - y_0 = d \cdot x$$

$$\therefore \quad d = \frac{y - y_0}{x}$$

The numerator $(y - y_0)$ is the vertical *increment* corresponding to the horizontal increment x. So d is the ratio of the increase along the y-axis corresponding to

a given increase along the x-axis. This is what we ordinarily mean by the slope or *gradient* of a line. When we say the gradient is 1 in 10, we mean that a vertical rise of 1 corresponds to a horizontal measurement of 10. If we say the gradient or slope of the line is 3 as in Chart 48, we mean that a vertical rise of 3 corresponds to unit horizontal shift.

Ex. 8.51. Plotting Graphs

Plot graphs for all the series given in Ex. 8.21. Write down the formula for each, using x and y.

*　　*　　*　　*　　*

The last exercise gave you practice in plotting points; but actually only two points are needed for plotting a *straight-line* graph. For accurate drawing, it is best to take two points as far apart as possible. It is often useful to take A_0 as one of the points, i.e. the value of y for which $x = 0$. Our formula is now:

$$y = y_0 + d \,.\, x$$

The second point on the graph can be found by giving x any convenient value, e.g. 5. Thus if

$$y = 8 + 3x$$

when $x = 5$,

$$y = 8 + 3 \cdot 5 = 23$$

We can thus plot $x = 0, y = 8$ as our first point, and $x = 5, y = 23$ as our second point. We then draw a straight line joining them and extending beyond them as far as we wish. For convenience we express these points thus:

$$(0,\ 8) \quad \text{and} \quad (5,\ 23)$$

Ex. 8.52. Graphs of Formulae

Change the formulae of Ex. 8.22 into the x, y notation and plot their graphs, using the scales stated.

Our remarks about *change of scale* for histograms apply equally to graphs. If you increase the y-scale the slope is steeper, if you increase the x-scale the slope is less steep.

Ex. 8.53

Construct the x, y formula for the series in Ex. 8.41, and then plot the graphs on each of the three given scales.

§ 6. APPLICATIONS OF THE STRAIGHT-LINE GRAPH

The two kinds of growth diagrams—the histogram and the graph—should strictly be used for two different kinds of quantities. Some quantities such as population, money or dots in a pattern can go up *only* by jumps. Thus a population is always a *whole* number of persons. Money is always a whole number of coins—even if they are only farthings or centimes or cents. Other quantities are not expressible by whole numbers. When we say that a table is 6 ft. wide we mean that 6 ft. is the scale division on our footrule nearest to one edge when we put the zero mark as nearly as possible in line with the opposite edge. Any measurement is an *estimate* of the nearest scale division. It never corresponds exactly to a whole number. Now measurements, e.g. baby's weight, go up steadily, not by jumps. Temperature rises and falls *smoothly*, and we can measure it to small fractions of a degree. It is this second kind of growth which the graph most truly represents, though it is sometimes *convenient* for the first kind, as when the units are too small for a satisfactory histogram.

Thus a graph gives us a picture of *continuous* growth, a histogram of growth *by jumps*. Whenever one continuous quantity is directly proportional to another we can draw a straight-line graph to show the relation, and not only to show it but to *measure the value of one corresponding to a given value of the other*. For example, Chart 49 shows how the readings of the same temperature measured on different thermometer scales, Centigrade and Fahrenheit, are connected. The F-readings are on the y-axis and the C-readings on the x-axis. Calling the F-readings T and the C-readings t, we have:

$$T = 32 + \frac{9}{5}t$$

$$(\text{compare } y = y_0 + d \,.\, x)$$

You see from this:

1. That the graph cuts the T-axis at 32° F. (i.e. 32° F. = 0° C.).
2. That the F-readings go up $1\frac{4}{5}$° or $1 \cdot 8$° for every 1° C.

Ex. 8.61. GRAPH-READING

From the graphs in Charts 49–50 find the following temperature-readings.

1. The F-reading for 5° C.
2. The F-reading for 25° C.
3. The F-reading for 90° C.
4. The F-reading for − 15° C.
5. The F-reading for 110° C.
6. The C-reading for 50° F.
7. The C-reading for 68° F.
8. The C-reading for 122° F.
9. The C-reading for − 4° F.
10. The C-reading for − 40° F.

*   *   *   *   *

The coefficient d of x in the equation of the straight-line graph always stands for the *slope*, if we define slope (or *gradient*) as the ratio of the vertical increment (change of y) corresponding to a given horizontal increment (change of x), or, what comes to the same thing, increase of y per unit increase of x. Whether the slope of a graph *looks* more or less steep to the eye depends on the scale. If the gradient of a railroad track is 1 in 10 ($d = 0 \cdot 1$), our graph gives us a faithful visual picture of the slope, when we lay out vertical and horizontal measurements on the *same* scale, e.g. 1 *in.* = 100 *yd.* upwards or forwards. If we use 1 in. to represent 10 yd. vertically and 100 yd. horizontally, the *visual* picture will be that of a track with a gradient of 1; but we can use it just as well to *read* off the height at a given horizontal distance from the beginning of the incline. When we use graphs as in the last exercise (Ex. 8.61), we can choose the *most convenient* scale for our purpose in either direction.

Chart 50 shows how we can make the Fahrenheit scale equal in length to the Centigrade by changing the scale of Chart 49. The graph then goes up at 45°, i.e. one vertical unit for every horizontal unit. This represents the readings on a *single* thermometer having *both* scales. The ratio of the F-unit to the C-unit is $5 : 9$ ($= 1 \cdot 0 : 1 \cdot 8$).

Ex. 8.62

Make graphs like that of Chart 50 to convert:

1. Pounds (£) into dollars (taking 1 *dollar* as 5*s*.).
2. Pounds (*lb.*) into kilograms (1 *lb.* = 0·45 *kilos*).
3. Litres into pints (1 *pint* = 0·57 *litres*).
4. Pints into cubic feet (1 *pint* = 0·02 *cu. ft.*).
5. Acres into square miles (640 *acres* = 1 *sq. mile*).
6. Centimetres into inches (1 *in.* = 2·54 *cm.*).
7. Miles into kilometres (1 *mile* = 1·61 *kilom.*).
8. Yards into fractions of a mile (1,760 *yd.* = 1 *mile*).
9. Litres into cubic feet.
10. Acres into square kilometres.

Ex. 8.63

From the graphs of Ex. 8.62 read off the following and *check* by arithmetic.

1. £5 15s. in *dollars.*
2. $4\frac{3}{4}$ lb. = ? *kilos.*
3. $2\frac{1}{2}$ pints = ? *litres.*
4. $\frac{3}{4}$ mile = ? *yds.*
5. $\frac{3}{4}$ sq. mile = ? *acres.*
6. 0·3 cu. ft. = ? *pints.*
7. $7\frac{1}{2}$ kilom. = ? *miles.*
8. 350 acres = ? *sq. kilom.*
9. 17 cm. = ? *inches.*
10. ? litres = 0·75 *cu. ft.*

* * * * *

The last exercises (8.62 and 8.63) illustrate the use of a graph as a *calculating device.* In all of them, we are measuring the *same characteristic* of a thing, e.g. its volume, its weight or its length, in *different sets of units*, and our graph show us how much growth in one system of units corresponds to the same increase in another system. When we plot a baby's weight (1 lb. = 1 vertical unit) month by month (1 month = 1 horizontal unit), we are measuring different characteristics (*weight* and *age*) of the same thing. So the units are necessarily different. A simple interest graph (*amount* of capital and *time*) is a *straight-line* graph of this sort. So also is a graph which shows how much we have to pay for a certain amount of coal. If we are told that coal is 4s. per *cwt.* or £4 per *ton*, we can *calibrate* our vertical and horizontal scales to read off the price in either shillings or pounds for either cwt. or tons. Suppose we represent price by vertical and weight by horizontal measurements. Let us make 1 large unit divided into 10 smaller ones on our square paper represent 10s. on the vertical and 10 cwt. on the horizontal scale. We can then *label* our horizontal scale in 2 columns thus:

Units	*0*	*1*	*2*	*3*	*4* . . .
Cwt.	0	10	20	30	40 . . .
Tons	0	0·5	1·0	1·5	2·0 . . .

We can now read off the price (vertical units) for either *cwt.* or *tons* directly, counting 1 small unit as 1 *cwt.* or 0·05 *tons* as required. Likewise we can label our vertical scale:

Units	*0*	*1*	*2*	*3*	*4* . . .
Shillings	0	10	20	30	40 . . .
Pounds	0	0·5	1·0	1·5	2·0 . .

On the vertical scale one small unit is either 1s. or £0·05. We can therefore use the same graph to read off the price in shillings of so many *cwt.* or so many tons or the price in £ of so many *cwt.* and so many tons. In fact we can label our

scale in as many units as we like. If 1 dollar is 5s., we can label our vertical scale in 3 columns as follows:

Shillings	0	1	2	3	4	5 . . .
Dollars	0	0·2	0·4	0·6	0·8	1·0 . . .
Pounds	0	0·05	0·1	0·15	0·2	0·25 . . .

Similarly we could label our horizontal scale in *lb.*, *cwt.*, *tons*, *kilos* simultaneously. To take advantage of this trick we have to use common sense when choosing our unit; and to use graph paper with a fine mesh. For instance, if we want to label a scale for pence or shillings, we might use graph paper on which 5 small units represent a penny and 6 large units represent a shilling.

Ex. 8.64

1. Make a graph showing the price in *pence* or *cents* of milk in *pints* and *litres* at 4d. per pint (1 dollar = 5s.).
2. If land is £50 per acre at an exchange rate of 4 dollars to the £, make a price graph showing the value of a given area measured in acres and sq. metres for sale in £ and $.

* * * * *

§ 7. SHAPE AND SCALE

We have now met three sorts of graphs:

(*a*) The *Contour-type*, e.g. height of a railroad track at a given horizontal distance from a fixed point, such as the beginning of the slope. Here we are measuring the *same characteristic* (a distance) of the same thing in the *same system of units.*

(*b*) The *Temperature-type*, e.g. degrees Centigrade and Fahrenheit or area in acres and square miles. Here we measure the *same characteristic* of the same thing in *different systems of units.*

(*c*) The *Price-type*, e.g. cost of coal per cwt. or baby's weight per month. Here we measure *different characteristics* (e.g. cost and weight or weight and age) of the same thing in units which are *necessarily different.*

All three types are serviceable as calculating devices; but the first may be something more. If we use the same scale for vertical and horizontal measurements it gives us a correct visual picture of what the graph stands for as well as pro-

viding us with a means of reading off the correct height at a given distance from a fixed point. If we use a larger horizontal unit, our straight-line graph will slope less steeply. If we use a smaller one it will slope more steeply. It is important to remember this, because we can use graphs not merely as calculating devices but as means for discovering a formula or equation connecting two sets of measurements. How we can use them with this end in view is another story; but it is well to realize in advance that what a graph looks like depends on *scale*, and what shape corresponds to what formula implies agreement about the scales we choose for our vertical and horizontal measurements.

If we measure the horizontal and vertical distances of successive points of any flat shape on the page of a book from the left bottom corner, we can reproduce it in miniature or magnify it by plotting the measurements on a finer or coarser mesh and joining the points. If the spacing of the mesh is equal in both directions, the shape is the same. Otherwise we distort it, as we distort a figure drawn on a sheet of rubber, when we stretch it in one or other direction. On a rainy day you can follow up the clues of Charts 51–52 by distorting figures in this way.

§ 8. THE VERNIER SCALE

A useful fact to know about scales is the basis of a device for making very precise measurements. Suppose we set two scales side by side (Chart 53), one marked in units of 12 in., the other below it in units of 11 in. The 3-unit mark of the lower lies opposite that of the upper when its zero mark is 33 in. to the left of the latter. Since the 3-unit mark on the upper is 36 in. from its own zero mark, the zero mark of the lower will be 3 in. in front of the zero mark of the one above it. If we set the two 4-unit marks level, the zero mark of the lower one will be 4 in. beyond the zero mark of the upper. Now suppose we have two 11-ft. poles, one marked out in 11 divisions 12 in. apart, the other in 12 divisions 11 in. apart. We put the first with its zero mark at one end of an object we want to measure. The other end of the object is somewhere between the 2nd and 3rd divisions. We know therefore that it is a little over 2 ft. long. Now bring the second pole alongside the first with its zero mark level with the end of the object. If the 4th division of the second pole lies level with the 4th division beyond the 2-ft. mark on the first we know that the end of the object is 4 in. from the 2-ft. mark, so that its correct measurement to the nearest inch is 2 ft. 4 in.

By combining two coarse scales we can therefore make fine measurements. In fact we can make measurements in terms of scale divisions separated by distances otherwise too small for the eye to distinguish without the help of the microscope. A combination of this sort is called a *Vernier* after its inventor. Chart 54

which shows it represents part of a ruler (magnified). The space from 2 to 3 represents 1 inch. The lower rectangle is the Vernier attachment which slides along the edge of the ruler. The distance to be measured is from the 0-end of the ruler (off the page) as far as the dotted line just beyond the second division after 2. Thus the measurement is a little more than 2·2 ins. The extra bit is x. To use the Vernier we slide it along until the edge is exactly under the dotted line and then look along the divisions till we see one which is just in line with a division on the ruler. It is the third division along.

Now study the Vernier itself. It is $\frac{9}{10}$ths of an inch long (magnified in the Chart) and divided into 10 parts. Thus each part is $\frac{9}{10}$ths of $\frac{1}{10}$th of an inch, i.e. $\frac{9}{100}$th or $\frac{10-1}{100}$ in. Starting from the dotted line we now have 2 series of steps, one on the ruler, each $\frac{1}{10}$th in., i.e. $\frac{10}{100}$ in., and the other one the Vernier, each $\frac{10-1}{100}$ in. They are out of step by the amount x. As we go along, the upper divisions gain $\frac{1}{100}$th on the lower divisions at each step. After 3 steps they are in line. Thus we know that they were out of step by $\frac{3}{100}$th in.

$$\therefore \quad x = \tfrac{3}{100} \text{ (inches)}$$

In this example they are out of step by 3 but the difference might be any number up to 9. Call it a. Then a steps on upper scale $= a$ tenths of an inch.

The Vernier units are $\frac{9}{10}$ths of the upper divisions. Thus a lower scale divisions $+$ distance $x = a$ upper scale divisions.

i.e.
$$\left(\tfrac{9}{10}a\right) + x = a$$
$$\therefore \quad x = a - \tfrac{9}{10}a$$
$$= \tfrac{1}{10}a$$

This is a device you can easily construct yourself from cardboard or plywood. You can mark it out in inches or centimetres or degrees or any other unit. It is useful whenever we want to measure to a degree of accuracy finer than our smallest distinguishable scale divisions, and is an essential part of all measuring instruments of high precision. The general rule is that we can measure a fraction $\frac{1}{x}$ of a unit on one scale if we have a second scale with divisions $\frac{1-x}{x}$ units apart. To measure minutes of an angle we need therefore a Vernier marked with divisions $\frac{59}{60}$ths of those on the scale of degrees.

CHAPTER 9

Graphical Representations and Solutions of Equations

§ 1. VARIABLES AND FUNCTIONS

Any number-sign used in Arithmetic, such as 8, 36, 1066, has a fixed meaning. We use it for referring to a *group of countable real things* such as eggs or men or stars. The number-signs which we used in Book I of this Algebra, such as n, T, F, stood for whole *series of numbers.* Thus they were more general than the numbers used in Arithmetic. For instance, T_n stood for the series of Triangular Numbers:

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \ldots$$

There are no values for Triangular Numbers in between these values. It is meaningless to ask for the rank of the Triangular Number whose value is 13. There is no triangular number between 10 and 15. Such numbers go up by jumps.

In this Book we are dealing with yet another kind of number. The numbers we have called x and y can have *any* values. We use them in dealing with quantities like temperature, weight, speed, etc., which do not vary by sudden jumps but smoothly, like a moving ship. The notion of keeping track of numbers which change smoothly like a moving ship arose naturally out of the practice of representing a ship's course on a grid of lines with which we are familiar as parallels of *latitude* and meridians of *longitude.* In fact the first mathematicians to try out this trick did so about the time—the fifteenth century—when the great navigations were under weigh. They called what we now call the *ordinates* (y guide-lines of the grid) *longitudines* and what we now call *abscissae* (x guide-lines of the grid) *latitudines.*

One name for quantities which vary smoothly like a moving ship is *variables.* Our graphs are diagrams showing the values of *variables.* The ones in this Book are all concerned with *two* variables. It is possible to have more than two but it is more difficult to deal with them. One variable x, which we call the *independent variable,* can have any values we choose to give it. The other variable y is fixed once we have fixed our value for x. Thus it depends on x, and so we call it the *dependent variable.*

What we call dependent and what we call independent is often a matter of taste; but in practical affairs the notion of *dependence* is important. A temperature-reading *depends* on the time. The speed of a car *depends* on its horse-power. The income-tax which you pay *depends* on your income. The weight of a baby *depends* on its age. *There are different kinds of dependence.* The great advantage of drawing graphs is that they show the kind of dependence at a glance. There is a special word to express the dependence of the dependent variable y on the independent variable x. We say y is a *function* of x. In symbols we write:

$$y = f(x)$$

The formulae in Book I were examples of functions of a special sort. The value of a figurate number depends on its rank, but of course the rank is always a whole number. Thus the formula represents *a discontinuous function*, one which increases by jumps. The formulae we use in this Book stand for *continuous functions*, ones which grow without interruption. So far we have met only one type illustrated by:

$$y = 8 + 3x$$
$$y = 32 + \tfrac{9}{5}x$$

The number series of Book I could be expressed by an equation (*formula*) or by a set of *patterns*. In this Book the functions we shall deal with can be expressed as an equation or as a *graph*. The *equation* enables us to *calculate* one variable for any value of the other. The *graph* enables us to *see* how the function varies, and—if we wish—to *measure* it instead of calculating it.

In our first two chapters we are dealing only with straight-line graphs. We call the functions of these graphs *linear functions*. They represent pairs of numbers in *Direct Proportion* to one another. Plotting the graph of a linear function is very simple. Squared paper is not necessary, if we have a set square to give us two directions at right angles. Choose any two values of the function, not too close together. These give you two points on the graph. Join them and there is your graph. Chart 55 shows that of $y = 3x + 5$.

Example: Plot the graph:

$$y = 4x - 6$$

Take two values of x, say -5 and $+5$.

$$x = -5 \qquad y = 4(-5) - 6 = -20 - 6 = -26$$
$$x = +5 \qquad y = 4(+5) - 6 = 20 - 6 = 14$$

Choose a scale which enables you to plot from -26 to $+14$ on the same page, e.g. 1 y-unit $= \frac{1}{10}$ inch.

The x-unit can be larger, e.g.:

$$1\ x\text{-unit} = \tfrac{1}{2}\ \text{inch}$$

Our two points are:

$$(x = -5, y = -26) \text{ and } (x = 5, y = 14)$$

We can write this more briefly:

$$(-5, -26) \text{ and } (5, 14)$$

As here, we shall always write the x-value first, and the y-value second. Mark the one point 5 x-units to the left of the y-axis and 26 y-units below the x-axis. Mark the other point 5 x-units to the right of the y-axis and 14 y-units above the y-axis. Then join the points.

By experience you learn in time to choose the most suitable units and to place the x- and y-axes in the most convenient part of the page.

Ex. 9.11. Plotting Linear Functions

Plot graphs of the following functions.

1. $y = 2x + 1$.
2. $y = 3x - 5$.
3. $y = -x + 2$.
4. $y = 5x - 4$.
5. $y = -10x + 6$.
6. $y = -8x - 12$.
7. $y = \frac{1}{2}x + \frac{3}{4}$.
8. $y = -\frac{1}{5}x - 2$.
9. $y = \frac{3}{10}x - 2\frac{1}{2}$.
10. $y = -\frac{1}{8}x + 8$.

N.B.—When the coefficient of x is fractional it is best to choose values of x which are multiples of the denominator, e.g. if coefficient is $\frac{1}{5}$ take, say, $x = 25$ or 40.

The graphs you have just drawn were miscellaneous, and not connected in any way. It is interesting to draw *families* of graphs (just as we drew *families* of numbers in Book I). You should draw each family on a single sheet of graph paper to bring out the family likeness.

Ex. 9.12. Families of Linear Functions

Draw each of the following sets of graphs on the same sheet and the same scale. (You may vary the scale from one family to another.)

1. $y = x$ $y = 2x$ $y = 3x$ $y = 4x$	2. $y = -x$ $y = -2x$ $y = -3x$ $y = -4x$
3. $y = x + 4$ $y = 2x + 4$ $y = 3x + 4$ $y = 4x + 4$	4. $y = x + 4$ $y = x + 3$ $y = x + 2$ $y = x + 1$
5. $y = 2x - 0$ $y = 2x - 1$ $y = 2x - 2$ $y = 2x - 3$	6. $y = x + 8$ $y = x + 5$ $y = x + 2$ $y = x - 1$
7. $y = -3x + 8$ $y = -3x + 0$ $y = 3x - 8$ $y = 3x - 16$	8. $y = 2x + 4$ $y = -2x + 4$ $y = 2x - 4$ $y = -2x - 4$
9. $y = x + 6$ $y = -x + 6$ $y = x - 6$ $y = -x - 6$	10. $y = \frac{1}{4}x + 2$ $y = \frac{2}{3}x + 2$ $y = \frac{3}{2}x + 2$ $y = 4x + 4$

When we studied families of numbers in Book I we were able to obtain a general formula for the whole family. We can likewise find a general equation for a family of graphs. In Chapter 8 we saw that the equation for *any* straight line can be written:

$$y = y_0 + dx$$

Mathematicians get into the habit of choosing a particular set of letters for a particular function and sticking to it. A common way of writing the above equation is:

$$y = mx + c$$

Here c is y_0 and m is d.

If you study carefully the graphs you have drawn in the last exercise you will see that:

(*a*) In some families m is fixed and c changes, e.g.:

$$y = 3x + 2 \qquad y = 3x + 4$$
$$y = 3x + 3 \qquad y = 3x + 5$$

We can write the family formula:

$$y = 3x + c$$

A family for which m is fixed, while c varies, is a set of parallel straight lines, i.e. lines with the same slope m as defined on pp. 176 and 177.

(*b*) In other families c is fixed and m changes, e.g.:

$$y = 2x + 5 \qquad y = 6x + 5$$
$$y = 4x + 5 \qquad y = 8x + 5$$

We can write the family formula: $y = mx + 5$.

A family for which c is fixed, while m varies, is a set of lines of varying slope all cutting the y-axis at the same point $y = c$.

(*c*) In some families of 2 members the numerical values of c and m are fixed, but the sign attached to c may be positive or negative, e.g.:

$$y = 2x + 4 \qquad y = 2x - 4$$

If the sign of c is positive the graph cuts the y axis above the level of the x-axis. If the sign of c is negative the graph cuts the y-axis below the level of the y-axis.

(*d*) In some families of 2 members the numerical values of c and m are fixed, but the sign attached to m may be positive or negative, e.g.

$$y = 2x \qquad y = 2x + 4$$
$$y = -2x \qquad y = 4 - 2x$$

When the sign attached to m is positive the graph slopes *upwards from left to right*. When it is negative the graph slopes *downwards* from left to right.

Ex. 9.13. Family Formulae

Find the family formula for questions 1–5 in Ex. 9.12.

§ 2. THE STANDARD LINEAR EQUATION

The standard equation $y = mx + c$ may appear in various disguises. It is generally advisable then to reduce it to the standard form.

Example:

$$3x + 4y + 5 = 0$$

We can write this:

$$4y = -3x - 5$$
$$\therefore\ y = -\tfrac{3}{4}x - \tfrac{5}{4}$$

Thus $m = -\frac{3}{4}$, and $c = -\frac{5}{4}$.

The equation $y = x$ would appear in standard form as:

$$y = 1x + 0$$

This has exactly the same meaning but it emphasizes that $m = 1$ and $c = 0$, i.e. the slope or gradient is 1, and the point of intersection on the y-axis is 0.

Occasionally, it is convenient to make x the dependent variable and y the independent. This is known as *changing the argument* or *changing the subject.*

Example:

$$y = 4x - 6$$
$$\therefore \quad 4x = y + 6$$
$$\therefore \quad x = \tfrac{1}{4}y + \tfrac{6}{4}$$

When we change the argument we must indicate that m and c have a different meaning. We can do this by calling them m' and c'. You can see from the above that:

$$m' = \frac{1}{m} \quad \text{and} \quad c' = -\frac{c}{m}$$

Thus we can write either

$$x = m'y + c' \qquad \text{or} \qquad x = \frac{1}{m}y - \frac{c}{m}$$

We shall call this the *Inverse Standard Form.*

Ex. 9.21. Reduction to Standard Form

Reduce the following equations first to standard form then to inverse standard form.

1. $10x - 5y + 3 = 0.$
2. $15y + 2x = 12.$
3. $-5 + 3x = -4y.$
4. $5(x + y) - 6(x - y) + 8(x + 2) = 0.$
5. $3x + 4y + 5(x + y) = 0.$

The quantity m is very important. As we have seen it gives us the slope or *gradient* of the graph. The gradient of a railway-line means the ratio of the vertical height through which you rise to the horizontal distance you traverse. You see little signs beside the line such as $\boxed{1 \text{ in } 120}$ or $\boxed{1 \text{ in } 384}$. Railway slopes are never very steep but the slope of a graph may be anything from 0 in 1 to 1 in 0. We write the gradient as a ratio or fraction, such as:

$$\frac{0}{1} \quad \frac{2}{3} \quad \frac{1}{10} \quad \frac{12}{5} \quad \frac{50}{3} \quad \frac{100}{1} \quad \text{etc.}$$

Now study the series:

$$\frac{1}{1} = 1 \qquad \frac{1}{\frac{1}{2}} = 2 \qquad \frac{1}{\frac{1}{4}} = 4 \qquad \frac{1}{\frac{1}{8}} = 8 \qquad \text{etc.}$$

You see that the smaller the denominator the larger the ratio. What happens as the denominator dwindles further still?

$$\frac{1}{\frac{1}{100}} = 100 \qquad \frac{1}{\frac{1}{100{,}000}} = 100{,}000 \qquad \frac{1}{0} = ?$$

We have no number to express $\frac{1}{0}$. How many times can you put your hand in your pocket and take nothing out? You can go on till the pocket is worn out. We call this *indefinitely large* number: *infinity*. The symbol for it is ∞. Thus:

$$\frac{1}{0} = \infty \text{ and } \frac{1}{\infty} = 0$$

As applied to graphs ∞ means the slope of a graph which is *perpendicular* to the x-axis. Thus when $m = \infty$ we have:

$$y = \frac{1}{0}x + c$$

0 and ∞ often give trouble in equations as they do not obey all the rules of ordinary numbers, and, of the two, ∞ is the more awkward. We may avoid it by interchanging x and y, using the *inverse standard form*:

$$x = m'y + c'$$
$$\therefore \; x = 0 + c'$$

This is the equation of the y-axis ($c' = 0$), and of all lines parallel to it.

Chart 56 gives you the equations for some special lines. Study it carefully and then answer the questions in the next exercise.

Ex. 9.22. Linear Functions of Special Interest. Charts 56–57

1. What are the gradients of A and B?
2. What is the value of c for A and B? Why?
3. What do you notice about D, B and E?
4. What do you notice about C, A and F?

5. Can you pick out any other families of 3 lines obeying the same rule as in No. 3 and No. 4?
6. What unusual feature do you notice in I and J?
7. Why is the x-axis labelled $y = 0x + 0$?
8. Why is the y-axis labelled $x = 0y + 0$?
9. Pick out all the graphs which have a gradient of 1 or -1. What do you notice?
10. What is the rule for the family C, D, E, F?

* * * * *

Now notice the following points:

(*a*) The x-axis itself is a graph, having as its equation:

$$y = 0 \qquad \text{i.e. } y = 0x + 0$$

(*b*) The y-axis is also a graph having as its equation:

$$x = 0 \qquad \text{i.e. } x = 0y + 0$$

(*c*) Any line parallel to the x-axis is a graph having as its equation:

$$y = c \qquad \text{i.e. } y = 0x + c$$

(*d*) Any line parallel to the y-axis is a graph having as its equation:

$$x = c' \qquad \text{i.e. } x = 0y + c'$$

(*e*) Hence *the lines of the squared paper are themselves all graphs*. They form two intersecting families, $y = c$ and $x = c'$.

(*f*) All parallel lines have the same gradient m.

(*g*) Perpendicular lines have negative inverse gradients, m and $-\frac{1}{m}$.

Verify the last conclusion by drawing pairs of graphs with negative inverse gradients, e.g. $y = 2x - 5$ and $y = -\frac{1}{2}x + 3$.

Ex. 9.23. Practice with Special Graphs

1. Draw the following graphs:

$$y = -4, \quad y = -2, \quad y = 0, \quad y = 2, \quad y = 4.$$

2. Draw the following:

$$x = -3, \quad x = -1, \quad x = 1, \quad x = 3.$$

3. Draw the graph $y = -2x + 6$. Where does it intersect the y-axis? Draw 3 more graphs, all parallel to this one, cutting the y-axis at $y = 3$, $y = 0$, $y = -3$. Write down their equations.
4. Draw the graph $y = \frac{1}{2}x - 2$. Draw four more graphs, all perpendicular to this one, and cutting the y-axis at $y = -4$, $y = -2$, $y = 0$, $y = 2$. Write down the equations of these four graphs.
5. Note the points at which each of the graphs you have drawn in Nos. 3 and 4 cut the x-axis. Substitute this value for x in the equation for each. What do you find?

§ 3. GRAPHICAL SOLUTION OF EQUATIONS

We can now connect the graphical view of Equations which we have reached in this chapter with the algebraic view of Book I, Chapter 7. Our aim in *solving* such equations was to find an unknown number. They were all puzzles of this type: *given certain information about n, find the value of n.* However complicated the information, we could always reduce it to a statement like this: so many times n + a certain number = 0. For example:

$$3n + 8 = 0$$

We solve this equation as follows: transfer 8 to the right-hand side and change its sign:

$$3n = -8$$

Divide by 3 (the coefficient of n):

$$n = -\frac{8}{3}$$

Thus $-\frac{8}{3}$ is the answer or *solution.*

Thus in solving the equation $3n + 8 = 0$ we are solely interested in finding what value of n makes the expression $3n + 8$ equal to 0. Any other values of n are not our concern. When we take the *graphical* view of this equation our interest is wider. We want to know the value of the expression $3n + 8$ for *all* values of n, or at any rate for a whole range of values. If a man has a motor accident a policeman will be interested in his doings at the time of the accident only, but a historian or biographer is interested in his whole life and career. The historical view is the graphical view. Thus when we adopt the graphical view we seek to represent the whole career of the expression $3n + 8$. We do this by representing

all its values on a graph for all values of n, over a certain range, from 0 upwards (and downwards if we wish, i.e. negative values). We are interested in the expression not merely as a quantity which equals 0 but as a variable quantity passing through a whole range of values, of which 0 is a particular case.

From this it follows that we can use a graph to solve an equation, i.e. to find the value of n (or x) for which the expression is equal to 0, but the graph gives us a lot more information as well. It shows us how the expression depends on n. Just now we used the term *function* to express this dependence, and since we are now dealing with continuous variables we shall speak of x rather than n.

Now where does y come into the picture? Well, we have a certain function, e.g. $3x + 8$. This function *may* be equal to 0. If so we write:

$$3x + 8 = 0$$

Because it is a variable function, it can take on a whole range of values such as:

$$3x + 8 = 1; \quad 3x + 8 = 2; \quad 3x + 8 = -5; \quad \text{etc.}$$

We need a symbol to express this range of values. This is the meaning of y. It means the complete range of values through which the function passes as x changes. The graph tells us the whole story of the function at a glance. Every point on the graph represents a certain value of x and the corresponding value of y. We call this pair of values the *co-ordinates of the point. The solution of the equation is given by the* x-*co-ordinate for which the* y-*co-ordinate is* 0.

Chart 57 indicates the graphical solution of the equation $2x + 3 = 0$. It also sums up a great deal of the information given in this chapter. The graph gives the solution of $2x + 3 = 0$ as $-1\frac{1}{2}$. We can verify this by algebra.

$$2x + 3 = 0 \qquad \therefore\ 2x = -3 \qquad \therefore\ x = -\tfrac{3}{2}$$

Ex. 9.31. Graphical Information

Make fully labelled charts similar to Chart 57 for each of the following equations. Check the graphical solution by algebra.

1. $y = -4x + 10$.
2. $y = \frac{2}{3}x - 5$.
3. $y = \frac{1}{10}x - 2$.
4. $y = -8x + 1$.
5. $y = -\frac{1}{5}x + 3$.

* * * * *

There is, of course, no need always to label graphs as fully as in Chart 57. The last exercise was intended simply to give you mastery of the language of graphs. Once you have mastered it you can take a good deal of it for granted.

In the next exercise you need to draw only the two axes and the graph itself. Give the values of the x- and y-co-ordinates along the axes and show clearly the solution. Write the equation along the graph itself.

Ex. 9.32. Graphical Solutions of Linear Equations

Solve the following equations by graph, reducing—where necessary—to standard form. Check solutions by algebra.

1. $y = 8x - 10$.
2. $4y - 5x = 12$.
3. $6x + 3y - 2 = 0$.
4. $\frac{3}{4}x - y + 8 = 0$.
5. $\frac{x}{2} + \frac{y}{3} + 1 = 0$.

By now you may have realized that finding the graphical solution is often slower and less accurate than finding the algebraic solution, and you may be wondering why anyone should take the trouble to solve equations by graph. The answer is that when we come to more complicated equations we shall find the graphical method is often the easier and sometimes the only one which gives us an answer. So it is best to master it by practising with simple equations first.

§ 4. SIMULTANEOUS EQUATIONS

At every point in its career the position of a graph can be labelled by two co-ordinates. At any point in its course, it cuts across one line parallel to the y-axis, distant a from the y-axis, and another line parallel to the x-axis and distant b from it. These two lines, whose equations are $x = a$ and $y =$ b, may or may not appear on the squared paper, but they can always be drawn if necessary. The x- and y-axes are themselves special examples. The *solution* of an equation is the point at which the graph cuts the x-axis, i.e. $y = 0$. But we are not always interested in this particular solution. We may want to know at what point our graph cuts some other line. Since every line represents a function this means that we want to find the point at which two functions are equal, i.e. when the graphs *cross* one another.

We have previously used the word *solution* to mean the value of x which makes $f(x) = 0$. We shall now use it to mean the values of x and y which make:

$$f_1(x) = f_2(x)$$

Here f_1 and f_2 are different functions. To distinguish these two sorts of solution we shall call the second kind the *simultaneous solution*. We shall call the two

equations *simultaneous equations*. Each function has its own graph. What concerns us is the point at which the two graphs intersect. Our simultaneous solution is given by the co-ordinates of this point.

Chart 58 shows you the simultaneous solution of the two equations

$$y = -2x + 4 \quad \textit{and} \quad 2y = -7 + x$$

In standard form these are:

$$y = -2x + 4 \quad \textit{and} \quad y = \tfrac{1}{2}x - \tfrac{7}{2}$$

The co-ordinates of the point of intersection are $x = 3$, $y = -2$. We can check that these values do, in fact, satisfy both equations.

$$-2x + 4 = -2(3) + 4 = -6 + 4 = -2$$

and

$$\tfrac{1}{2}x - \tfrac{7}{2} = \tfrac{1}{2}(3) - 3\tfrac{1}{2} = 1\tfrac{1}{2} - 3\tfrac{1}{2} = -2$$

Note that the graph $y = -2x + 4$ cuts the x-axis at the point $x = 2$. If we regard the x-axis as simply another graph, the graph of $y = 0$ in fact (see Chart 56), we see that we can regard our first use of the term *solution* as simply a special case of the second. *The solution of a single equation is in fact the simultaneous solution of that equation with the equation* $y = 0$. It is worth while to think hard about this. It is a good example of something we are continually finding in mathematics. We use a term in a certain sense. We explore its use to see where it gets us. We discover new things. We then find ourselves compelled to use the term in a wider sense; but the wider meaning includes the narrower meaning as a *special* case.

Ex. 9.41. Simultaneous Graphical Solutions

Find the simultaneous solutions of the following pairs of equations. (Reduce to standard form, where necessary. Check by substitution.)

1. $y = \tfrac{1}{2}x + 4$
   $y = -\tfrac{3}{2}x + 8$
2. $2x + y + 2 = 0$
   $3y + 2x - 6 = 0$
3. $x + 4y + 28 = 0$
   $5x - 2y + 8 = 0$
4. $4(x + 2y + 3) = 12$
   $2(x + y + 3) = 0$
5. $4x + 3y = 3$
   $-6y + x = -3$
6. $2(6x + 2y + 5) = 10$
   $2(2x + 6y - 5) = -10$
7. $2(3y - 2x) = 5 - 4x - 35$
   $3(2y + 3x) = 5 + 6y + 22$
8. $x + 5y + 1 = 0$
   $3x - 2y - 1 = 0$
9. $24y - 8x - 3 = 0$
   $24y - 32x + 15 = 0$
10. $3(x + y + 1) - 2(x + y + 5) = -5$
    $2(x - y + 1) - 3(x - y + 5) = -21$

§5. ALGEBRAIC METHOD FOR SIMULTANEOUS EQUATIONS

In Book I we dealt with simple equations as a method of solving certain number-problems or puzzles. In this Book we have treated them as functions to be represented as graphs. We have started off by treating simultaneous equations as graphs. We shall now deal with them algebraically. Suppose we have a puzzle of this type: the sum of two numbers is 12 and their difference is 6, find the number.

Write n for the greater number, m for the smaller number.

$$\text{Sum: } n + m = 12.$$
$$\text{Difference: } n - m = 6.$$

Neither of these equations can be solved alone. They must be solved *simultaneously*. This does not mean *at the same moment*. It means solving one with the help of the other, so as to obtain the same answer for both. There are two algebraic rules to help us:

1st. We can solve only one equation at a time.

2nd. We can do what we like to an equation so long as we treat both sides in the same way.

We cannot find both unknowns at once. So we must get rid of one for the time being. This is called *elimination*. Our two equations are:

$$n + m = 12$$
$$n - m = 6$$

If we add the two equations together we have $(n + n) + (m - m) = 12 + 6$, i.e. $2n + 0 = 18$.

$$\therefore\ 2n = 18$$
$$\therefore\ n = 9$$

We can find m in several ways.

(1) Repeat the first process, but this time eliminating n. To do this we must *subtract* one equation from the other.

$$\begin{array}{r} n + m = 12 \\ \underline{n - m = \ \ 6} \end{array}$$

$$\therefore\ (n - n) + m - (-m) = 12 - 6$$

$$n - n = 0 \qquad m - (-m) = +2m \qquad 12 - 6 = 6$$

$$\therefore\ 2m = 6 \qquad \therefore\ m = 3$$

(2) Replace n in the first equation by the value 9 first found:

$$9 + m = 12$$
$$m = 12 - 9 = 3$$

(3) Replace n in the second equation (i.e. *substitution*):

$$9 - m = 6$$
$$-m = 6 - 9 = -3 \qquad \therefore \quad m = 3$$

We can choose whichever method is simplest for the given figures.

Next take this puzzle: I have two kinds of coin. Four of the first and eight of the second give me £1. Eight of the first and four of the second give me 16s. What are the coins?

Let the value of the first coin be x. Let the value of the second coin be y. Our equations are:

$$(1) \quad 4x + 8y = 20$$
$$(2) \quad 8x + 4y = 16$$

We could eliminate y by subtraction if we had $8y$ in the second equation. We can get this by multiplying *both sides* of the second equation by 2.

$$(1) \quad 4x + 8y = 20$$
$$(2) \quad 16x + 8y = 32$$
$$\therefore \quad -12x = -12$$
$$\therefore \quad x = 1$$

Now substitute in (1):

$$4 \cdot 1 + 8y = 20 \qquad \therefore \quad 8y = 20 - 4 = 16$$
$$\therefore \quad y = \frac{16}{8} = 2$$

Answer: 1s. and 2s.

Here is another favourite type of puzzle. A man is 6 times as old as his son this year. Next year he will be 7 times as old as his son was last year. Find their ages.

Let $f =$ age of father; $s =$ age of son.

Then $f = 6s$ and $f + 1 = 7(s - 1)$

Before we can eliminate and substitute we must arrange these more conveniently:

$$f - 6s = 0$$
$$f + 1 - 7(s - 1) = 0$$
$$f + 1 - 7s + 7 = 0$$
$$f - 7s = -8$$

So our pair of equations is:

$$f - 6s = 0$$
$$f - 7s = -8$$
$$\therefore \quad +s = +8 \quad \text{(by } \textit{subtraction}\text{)}$$
$$\therefore \quad f = 48 \quad \text{(by } \textit{substitution}\text{)}$$

Some equations may involve fractions. Here is an old Greek problem from Maurice Kraitchik's entertaining book *Mathematical Recreations.*

A: "I have what B has and the third of what C has."

B: "I have what C has and the third of what A has."

C: "And I have ten *minae* and the third of what B has."

How many *minae* did each have? The fact that you may not know what *minae* are does not matter.

Let a stand for A's share; b stand for B's share; c stand for C's share.

$$\therefore \quad a = b + \frac{c}{3} \quad b = c + \frac{a}{3} \quad c = 10 + \frac{b}{3}$$

We can start by eliminating c. To do this put $\left(10 + \frac{b}{3}\right)$ in place of c in the first two equations and simplify:

$$(1) \quad a = b + \frac{1}{3}\left(10 + \frac{b}{3}\right)$$

$$(2) \quad b = \left(10 + \frac{b}{3}\right) + \frac{a}{3}$$

Now we can write:

$$(1) \quad a = b + \frac{10}{3} + \frac{b}{9}$$

Multiply by 9 to remove fractions:

$$9a = 9b + 30 + b$$
$$\therefore \quad 9a - 10b = 30$$

Next treat equation (2) in the same way:

$$3b = 30 + b + a$$
$$\therefore \quad -a + 2b = 30$$

Now put the two equations together:

$$(1) \quad 9a - 10b = 30$$
$$(2) \quad -a + 2b = 30$$

Multiply (2) by 5:

$$(1) \quad 9a - 10b = 30$$
$$(2) \quad -5a + 10b = 150$$

Add the two equations. The $10b$s drop out, i.e.:

$$4a = 180$$
$$\therefore \quad a = 45$$

Substitute in (2):

$$-a + 2b = 30$$
$$\therefore \quad -45 + 2b = 30$$
$$\therefore \quad 2b = 30 + 45 = 75$$
$$\therefore \quad b = \frac{75}{2} = 37\tfrac{1}{2}$$

Also from (3):

$$c = 10 + \frac{b}{3} = 10 + \frac{37\frac{1}{2}}{3}$$
$$= 10 + \frac{75}{6} = 22\tfrac{1}{2}$$

Answer: A had 45 minae; B had $37\frac{1}{2}$ minae; C had $22\frac{1}{2}$ minae.

Ex. 9.51. Easy Simultaneous Equations

Solve the following equations by algebra. Check your answers by graph.

1. (*a*) $4x + 3y = 24$, $3x + 3y = 21$
   (*b*) $6x + 10y = 40$, $x + 10y = 15$
   (*c*) $7x + y = 14$, $5x + y = 12$
   (*d*) $12x + 12y = 24$, $x + 12y = 13$

2. (*a*) $10n + 4m = 72$, $5n + 8m = 54$
   (*b*) $4n + 2m = 42$, $n + 4m = 28$
   (*c*) $12n + m = 34$, $20n + 3m = 70$
   (*d*) $4n + 6m = 110$, $3n + 2m = 45$

3. (*a*) $6p - 3q = 36$, $12p + q = 44$
   (*b*) $3p - 6q = -42$, $10p + 5q = 10$
   (*c*) $-8p - 7q = 9$, $-6p + 10q = -40$
   (*d*) $-10p + 4q = -10$, $-5p + 6q = -45$

4. (*a*) $5u + 3v = 10$, $6u + 4v = 0$
   (*b*) $4u + v = 0$, $-6u - 2v = -30$
   (*c*) $-4u + 5v = -120$, $8u - 7v = 0$
   (*d*) $7u - 3v = 0$, $-14u + 6v = 0$

5. (*a*) $8r - 10s = 40$, $-5r + 2s = -25$
   (*b*) $12r + 10s = 100$, $-3r - 3s = -30$
   (*c*) $-6r - 6s = 6$, $15r + 5s = -5$
   (*d*) $7r - 6s = 0$, $6r - 7s = 0$

Ex. 9.52. Harder Simultaneous Equations

Solve these by graph. Check your answers by algebra.

1. $2(3z - 4) = 5(4w - 14)$
   $6(2z + 3) = 9(5w - 14)$
2. $4(10z - 3) = 7(5w - 26)$
   $8(3z + 8) = 11(w + 2)$
3. $5(5z - 13) = -3(2w - 9)$
   $-7(4z - 16) = -8(w - 14)$
4. $-2(4z + 4) = 8(8w - 10)$
   $10(-10z - 10) = -20(5w + 5)$
5. $8(2h + 5k) = 4(h + 4k)$
   $2(3h - 40) = 4(3k + 10)$
6. $12(6h - 3k) = -6(3h + 6k)$
   $7(5h + 3) = -3(4k - 3)$
7. $12(h + k) = 8(4h - k)$
   $2(10k - 2) = -4(10 - 2k)$
8. $-2(3h - 10k) = 6(5k - h)$
   $7(30 + h) = 5(7 - 10k)$
9. $\dfrac{3f}{2} + 4g + 1 = 12g$
   $\dfrac{5f}{6} + 7g + \tfrac{1}{2} = \dfrac{20g}{3}$
10. $\dfrac{30f}{7} - \dfrac{15g}{14} + \dfrac{1}{7} = 1$
    $\dfrac{-20f}{11} + \dfrac{50g}{33} - \dfrac{5}{22} = \dfrac{-10f}{66}$

Ex. 9.53. Simultaneous Equations including Three Variables

Solve the following algebraically. Check by substitution. (You cannot use the graphical method with three variables.)

1. $\dfrac{5x+3y+2}{8} = \dfrac{2x+y}{32}$

   $\dfrac{7x+y}{16} = \dfrac{x-y-12}{10}$

2. $\dfrac{3x+7y}{11} = \dfrac{11x+37}{7}$

   $\dfrac{100(x-y)}{3} = 10 - \dfrac{1}{3}$

3. $6j + 3k = 12$

   $5k + 4l = 22$

   $3l + 2j = 11$

4. $10a + 11b + 12 = 1$

   $9b + 10c + 11 = 12$

   $8c + 9a + 10 = 18$

5. $10u + v + \dfrac{w}{10} = 3$

   $10v + w + \dfrac{u}{10} = \dfrac{2001}{100}$

   $10w + u + \dfrac{v}{10} = \dfrac{501}{5}$

6. $6p + 3q = 5r + 15$

   $r + 3p = 6q + 30$

   $q + 3r = 12p + 5$

7. $\dfrac{8}{15}(H+L) = W + \dfrac{1}{15}$

   $\dfrac{12}{11}(L+W) = 1 - \dfrac{8H}{11}$

   $\dfrac{6}{5}(W+H) = 8L - 1$

8. $\dfrac{X}{5} + \dfrac{2Y}{3} - \dfrac{3Z}{2} = 6$

   $\dfrac{X}{2} - \dfrac{Y}{2} - \dfrac{Z}{2} = 0$

   $\dfrac{X}{10} + \dfrac{Y}{6} + \dfrac{Z}{8} = -3$

9. $\dfrac{5}{12}E + \dfrac{3}{5}K = \dfrac{Q}{2} + 4$

   $\dfrac{7}{8}E + \dfrac{3}{20}K = \dfrac{50}{14}$

   $\dfrac{E}{6} + \dfrac{K}{12} = \dfrac{Q}{7} - 3$

10. $16\dfrac{(A+B+C)}{3} = 1$

    $\dfrac{4(8A+4B+2C)}{41} = \dfrac{1}{2}$

    $\dfrac{2(2A+6B+10C)}{25} = \dfrac{1}{4}$

CHAPTER 10

Graphs of Quadratics

§ 1. LINEAR AND QUADRATIC FUNCTIONS

The graphs of Chapters 8 and 9 were all *straight-line* graphs. They had the same *shape*, differing only with respect to *position* and *slope*. In this chapter we shall study a family of functions represented in graphical form by the same characteristic type of *curved* line, called a PARABOLA. In contradistinction to *linear* functions, which we have studied in the two preceding chapters, we call this family the *quadratic* functions. Just as we can have linear functions (i.e. Arithmetic Progressions) which grow by jumps, and linear functions which grow continuously, we can also have quadratic functions of either type. The linear function is one which involves only the *first power* of the independent variable which we represent by n, if it must be a whole number, or by x if it can grow by any increment, however small. We shall start by comparing linear and quadratic functions of the *discontinuous* type. The general formula of the linear function is then:

$$F_n = F_0 + dn$$

Here F_n is the nth term after the initial (o*th*) one. This formula contains two fixed numbers characteristic of the particular linear function (*Arithmetical Series*) for which it stands. Such a fixed number is called a *constant*. The constant F_0 is the numerical value of the *initial* term and d is the numerical value of the common difference. We can visualize a discontinuous numerical function by a hollow pattern as shown on pp. 100–1 (Book I) or by the equal-step histogram of p. 168, Chapter 8. The latter has the advantage of bringing two characteristics of arithmetic series into sharp relief:

(*a*) values of F_n may be either *negative* or *positive*, negative values being represented by columns *below*, positive values by columns *above* the base line;
(*b*) successive terms differ by equal steps.

The linear function has *two* constants and contains only the *first* power of n. The constants may have any whole-number values. If $F_0 = 0$ and $d = 1$, the series so defined is the series of *natural* numbers. If $F_0 = 0$ and $d = 2$ we have the *even* numbers. If $F_0 = -1$ and $d = 2$, we get the *odd* numbers. Most of the

formulae of Book I refer to series which contain the second power of n, i.e. n^2. They are then *quadratic* functions, if they contain no *higher* power. The general formula of a quadratic function contains 3 constants, and is:

$$F_n = F_0 + an + bn^2$$

When $F_0 = 0 = a$ and $b = 1$, we have the squares of Chart 3, i.e.:

$$Q_n = n^2$$

When $F_0 = 0$ and $a = \frac{1}{2} = b$, we have the triangular number series:

$$T_n = \tfrac{1}{2}n + \tfrac{1}{2}n^2 = \tfrac{1}{2}n(n + 1)$$

Ex. 10.11

Write down the values of the constants F_0, a and b for:

1. The pentagonal numbers of Chart 21.
2. The hexagonal numbers of Chart 21.
3. The stellate numbers of Chart 23.

* * * * *

Chart 59 shows the histogram of Q_n (the *squares* of the natural numbers). In accordance with the law of signs $(-n)(-n) = n^2 = (+n)(+n)$. Hence we can make a table like this:

n	...	-5,	-4,	-3,	-2,	-1,	0,	$+1$,	$+2$,	$+3$,	$+4$,	$+5$ ...
Q_n	...	25,	16,	9,	4,	1,	0,	1,	4,	9,	16,	25 ...

Our histogram brings out two differences between *linear* and quadratic functions:

(*a*) all values of a quadratic function lie above a base-line and are thus *positive*,

(*b*) a quadratic function grows by *increasing steps*.

If we make a difference table showing the steps (ΔQ_n) by which Q_n increases, the steps $(\Delta^2 Q_n)$ by which ΔQ_n increases and so on, it looks like this:

Q_n	... 25	16	9	4	1	0	1	4	9	16 ...
First difference (ΔQ_n) ...	-9	-7	-5	-3	-1	1	3	5	7	...
Second difference $(\Delta^2 Q_n)$ ..	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$	$+2$		...
Third difference $(\Delta^3 Q_n)$ ..	0	0	0	0	0	0	0			...

If you make a new histogram of the steps, as in Chart 64, you will see that the first differences form an arithmetic series, or linear function. Consequently, the second differences are constant like the steps of a linear function and the third ones vanish. This is true of any quadratic function.

Ex. 10.12

Construct numerical series from $n = -5$ to $n = +5$ for the following quadratic functions. Make histograms based thereon, and construct a *difference table* like the above for each. Choose a vertical scale suitable to each.

1. $F_n = 5n^2$.
2. $F_n = n(n + 2)$.
3. $F_n = \frac{1}{2}n(n + 1)$.
4. $F_n = (n + 1)(n + 3)$.
5. $F_n = (n + 10)(n - 10)$.

* * * * *

Having satisfied yourself that all these quadratic functions increase by steps which make up an arithmetic series, you can also satisfy yourself that this is necessarily so. By the definition given above:

$$\begin{aligned} F_n &= F_0 + an + bn^2 \\ \therefore\ F_{n+1} &= F_0 + a(n + 1) + b(n + 1)^2 \\ &= F_0 + (a + b) + (a + 2b)n + bn^2 \\ \therefore\ F_{n+1} - F_n &= (a + b) + 2bn \end{aligned}$$

Since a and b are both constants we can replace this sum by another constant C_1, and since b is a constant $2b$ is also a constant, which we can write as C_2, so that

$$F_{n+1} - F_n = C_1 + C_2n$$

This has 2 constants and no power of n higher than the first. In other words, the formula for the difference between successive terms of any quadratic function is the formula of an A.P. We can thus make a third distinction between a linear and quadratic function, namely: *linear functions increase by equal steps and quadratic functions increase by steps which make up an A.P., i.e. themselves increase by equal steps.*

* In the notation of the table on p. 203 we can define a linear equation thus:

$$\Delta^1 . F_n = C; \quad \Delta^2 F_n = 0$$

For a quadratic function the corresponding *difference* equations are:

$$\Delta^1 F_n = C_1 + C_2n; \quad \Delta^2 F_n = C_2; \quad \Delta^3 F_n = 0$$

§ 2. THE PARABOLA

If we join the mid-points of the top ends of the columns of our equal-step histogram for an A.P., we get a straight line. If we join the mid-points of the top ends of the columns of our quadratic histogram of Chart 59 we get a succession of straight lines, each more steep than its successor, if to the left of the zero mark, and more steep than its predecessor, if to the right of it. Joining the mid-points in this way takes no cognisance of what happens when the independent variable is not a whole number; and when we pass from the histogram of Chart 59 to the graphs of Charts 60–61, we are no longer concerned with whole numbers alone. A graph represents a continuous function. That is to say, the independent variable may have any value between consecutive whole numbers. Thus we might plot the following points:

x	. . .	$\frac{1}{2}$,	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	3	. . .
y	. . .	$(\frac{1}{2})^2$,	$(1)^2$,	$(1\frac{1}{2})^2$,	$(2)^2$,	$(2\frac{1}{2})^2$,	$(3)^2$	. . .
i.e.	. . .	$\frac{1}{4}$,	1,	$2\frac{1}{4}$,	4,	$6\frac{1}{4}$,	9	. . .

We can go on filling in the intervening gaps as long as we like. In this way we get a series of points which merge indistinguishably into the curved outline of Chart 60. As an exercise, draw the graph of Chart 60, filling in the above points. Take 2 squared-paper units (s.p.u.) as your unit for x and y. Then each whole number is a multiple of 2 s.p.u. Each $\frac{1}{2}$ value is 1 s.p.u. and each $\frac{1}{4}$ is $\frac{1}{2}$ s.p.u. You can judge the $\frac{1}{2}$ s.p.u. by eye.

Plotting straight lines is very simple. Plotting curves is tricky. This is due not so much to the difficulty of joining points by a smooth curve as to the difficulty of accurately plotting the points. The beginner is apt to get lost in a maze of squares, scales, units, decimals, etc., and this takes most of the fun out of graph-drawing. The mistake lies in using graph-paper which is too fine. Paper ruled in millimetres or $\frac{1}{10}$th inch squares is very useful for exact scientific work but very bewildering at an elementary stage. It is also bad for the eyes. The ideal paper is that used widely on the Continent, ruled in $\frac{1}{2}$ cm. squares, with no finer divisions and no thicker lines. The $\frac{1}{4}$-inch squared paper commonly used for junior arithmetic books in this country is also quite suitable. A fairly convenient size can also be obtained by ruling vertically down an ordinary ruled page. Here the lines are usually about $\frac{1}{3}$ inch. With all these it is easy to judge the half- and quarter-values, which are fine enough for most purposes. Unless we state otherwise we shall assume that one of the above sizes is being used. All sub-divisions can be made by eye. The rulings on our charts are settled by printing requirements and are not necessarily the same throughout the book.

We have seen (p. 188) that the coefficient m in the linear function $mx + c$ measures the steepness of the line, if we keep to the same scale for x and y measurements. We can make the line more steep or otherwise by changing the

scale of either. The following exercise will help you to see the meaning of the constant m, and its relation to the scale used, in the quadratic

$$y = mx^2$$

Ex. 10.21. Effect of Scale

Plot the graphs of the following functions, each twice over, using the two y-scales shown. Take x-unit as 1 s.p.u. throughout.

1. $y = 2x^2$.	Scale 1.	1 y-unit = 1 s.p.u.
	Scale 2.	1 y-unit = $\frac{1}{2}$ s.p.u.
2. $y = \frac{1}{4}x^2$.	Scale 1.	1 y-unit = 1 s.p.u.
	Scale 2.	1 y-unit = 4 s.p.u.
3. $y = 5x^2$.	Scale 1.	1 y-unit = 1 s.p.u.
	Scale 2.	1 y-unit = $\frac{1}{5}$ s.p.u.
4. $y = \frac{1}{5}x^2$.	Scale 1.	1 y-unit = 1 s.p.u.
	Scale 2.	1 y-unit = 5 s.p.u.
5. $y = 10x^2$.	Scale 1.	1 y-unit = 1 s.p.u.
	Scale 2.	1 y-unit = $\frac{1}{10}$ s.p.u.

From the last exercise you will have realized that we have been dealing with what is really one and the same curve throughout but the coefficient of x^2 has the effect of elongating it or flattening it. By a change of scale we can always restore it to the original shape of Chart 61. Let us now examine another set of quadratics (Chart 62), introducing a second constant, i.e.:

$$y = mx^2 + c$$

Ex. 10.22

With scales (*a*) 1 y-unit = 1 s.p.u.; (*b*) 1 y-unit = $\frac{1}{2}$ s.p.u. draw the following:

1. $y = 2x^2 - 3$.
2. $y = 2x^2 - 1$.
3. $y = 2x^2 + 1$.
4. $y = 2x^2 + 2$.

With scales (*a*) *as before*; (*b*) 1 y-unit = 4 s.p.u.:

5. $y = \frac{1}{4}x^2 - 4$.
6. $y = \frac{1}{4}x^2 - 1$.
7. $y = \frac{1}{4}x^2 + 1$.
8. $y = \frac{1}{4}x^2 + 3$.

The last set of examples shows that the addition of a second constant c does not affect the shape of the curve, but only its position relative to the y-axis. With appropriate use of scale, the same parabola $y = x^2$ drawn with 1 s.p. unit

on both the y-scale and the x-scale therefore serves for any quadratic of the type $y = mx^2 + c$, if we apply the following rules:

(*a*) choose the x- and y-units so that 1 s.p.u. is 1 x-unit and 1 s.p.u. is m y-units;

(*b*) place the standard curve on the grid so that it cuts the y-axis symmetrically at the point $y = c$. If c carries a minus sign this means that it cuts the y-axis c y-units *below* the base-line.

We can thus graph a large number of quadratic functions by placing one and the same standard parabola ($y = x^2$) on our grid with due regard to these two rules. It is therefore convenient to have a stencil for the purpose like Chart 61. Make one for yourself as follows:

Rule x- and y-axes on squared paper, putting the x-axis near the bottom of the page. Take 2 s.p.u. as unit for both x and y. Graduate x-axis from -4 to $+4$ and y-axis from 0 to 20. Plot the following points:

x	0	$\pm\frac{1}{2}$	± 1	$\pm 1\frac{1}{2}$	± 2	$\pm 2\frac{1}{2}$	± 3	$\pm 3\frac{1}{2}$	± 4
y	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9	$12\frac{1}{4}$	16

Mark each with a fine pencil-point and join up to form as smooth a curve as possible. Now paste the squared paper on a piece of moderately stiff cardboard. Cut along the curve and straight across the top at $y = 16$, with a razor-blade or sharp penknife. You now have a parabolic stencil.

The quadratic function of Ex. 10.22 had 2 constants. The general quadratic function has three. Its equation has the form:

$$y = mx^2 + px + c$$

As an example plot the following of which $m = 1$, $p = -8$ and $c = +20$:

$$y = x^2 - 8x + 20$$

To do so, it is necessary to make a table such as the following, and to draw a smooth curve through the corresponding values for x and y:

x	0	1	2	3	4	5	6	7
x^2	0	1	4	9	16	25	36	49
$-8x$	0	-8	-16	-24	-32	-40	-48	-56
$+20$	20	20	20	20	20	20	20	20
y	20	13	8	5	4	5	8	13

If you place your parabolic stencil on the curve so obtained, you will find that *it fits*. The only result of adding a term of the general type px is that it displaces the origin with reference to the x-axis. You will get more light on this by studying the family of parabolas in Chart 62. By substitution, verify that each equation satisfies one or two points on each curve. The curve (C) passes through the points $x = 3, y = 5$. So substitute these values in the equation:

$$y - 4 = (x - 4)^2$$

You then get:

$$5 - 4 = (3 - 4)^2 = - 1^2 = 1$$

Now study the family equation:

$$y - e = (x - d)^2$$

You then see that:

(*a*) the constant e measures the distance of the *vertex* (lowest point) of the parabola from the x-axis and along the y-axis;

(*b*) the constant d measures its distance from the y-axis and along the x-axis. Thus the effect of putting in the constants e and d is simply to shift the *origin* of the standard parabola which answers for the equation $y = x^2$ by a *horizontal* distance d and a *vertical* distance e. Now multiply out. You then get, instead of the above:

$$y - e = x^2 - 2dx + d^2$$

$$\therefore \quad y = x^2 - 2dx + d^2 + e$$

Since d and e are constants we can replace $- 2d$ by another fixed number p and $(d^2 + e)$ by another fixed number c, so that:

$$y = x^2 + px + c$$

This is the equation of the general quadratic. Thus the parabola stencil will do the work of plotting the curve of any quadratic, when you have reduced it to the form given above. We need not make a table like the one just given for:

$$y = x^2 - 8x + 20$$

Instead of doing so, we first express the right-hand side as a perfect square (see Book I, Ch. 7). To do this we must add the square of half the coefficient of x, viz. $(\frac{8}{2})^2$, i.e. 16.

$$y + 16 = (x^2 - 8x + 16) + 20$$

Transpose the 20 and express the right-hand side as a square:

$$y - 4 = (x - 4)^2$$

Then $c = 4$ and $d = 4$. Thus the new origin (where the y- and x-axes cross) is at the point (4, 4) and we plot our parabola from there. We call this process: *locating the vertex.*

Ex. 10.23. Change of Origin

Plot the parabola $y = x^2$ with the following points for origin and write the new equation for each, in both the familiar and the disguised form:

1. $(+2, +2)$
2. $(+1, -3)$
3. $(-3, +5)$
4. $(-2, -1)$
5. $(0, -6)$

Ex. 10.24. Locating the Vertex

Change the following equations to the familiar form and so locate the vertex. Then plot the graph.

1. $y = x^2 - 8x + 17$
2. $y = x^2 + 12x + 34$
3. $y = x^2 - 5$
4. $y = x^2 + 6x + 9$
5. $y = x^2 - 4x - 2$

* * * * *

§ 3. THE GENERAL QUADRATIC

We can write the general quadratic equation:

$$y = ax^2 + bx + c$$

where a, b, c can have any numerical values. We then locate the vertex as follows:

1. Divide through by a:

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

2. Add the square of half the coefficient of x to both sides:

$$\frac{y}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a}$$

$$\therefore \quad \frac{y}{a} + \frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$$

$$\therefore \quad \frac{y}{a} + \frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

If we put $aY = y$ we have:

$$Y + \frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

Now if $Y =$ 1 s.p.u., $y = a$ s.p.u. So this is equivalent to making 1 s.p.u. on the x-axis equivalent to a s.p.u. on a new y-scale. We can now put:

$$\frac{b^2 - 4ac}{4a^2} = -E; \quad \frac{b}{2a} = -d$$

$$\therefore \quad Y - E = (x - d)^2$$

We can now plot with the stencil for $y = x^2$ equations like this one:

$$y = 3x^2 - 5x - 7$$

To do so, we have to change both scale and origin.

(*a*) *Scale.* Adopt a new scale Y on which the units are such that $3Y = y$, and make the x-units equal to the Y-units. Draw both the y and the Y-scales side by side as in Chart 63. Now we have:

$$3Y = 3x^2 - 5x - 7$$

Divide through by 3.

$$Y = x^2 - \frac{5}{3}x - \frac{7}{3}$$

Notice carefully what the substitution $3Y = y$ means, because we shall often have to rely on this trick. To say that $3Y$ is equivalent to y means that 3 units on the y-scale is equivalent to 1 unit on the new Y-scale. That is to say the unit on the Y-scale is 3 *times as great as* the unit on the old scale. More generally, if $y = nY$, we may say that 1 s.p.u. on the Y-scale corresponds to n s.p.u. on the y-scale, or that the value of 1 s.p.u. on the Y-scale is equivalent to n s.p.u. on the y-scale.

(*b*) *Origin.* Complete the square as in § 2.

$$Y + \left(\frac{5}{6}\right)^2 = x^2 - \frac{5}{3}x + \left(\frac{5}{6}\right)^2 - \frac{7}{3}$$

$$Y + \frac{25}{36} + \frac{7}{3} = \left(x - \frac{5}{6}\right)^2$$

$$Y + \frac{25 + 84}{36} = \left(x - \frac{5}{6}\right)^2$$

$$Y + \frac{109}{36} = \left(x - \frac{5}{6}\right)^2$$

Comparing this with

$$y - e = (x - d)^2$$

we see that the new origin is at

$$\left(d = \frac{5}{6},\quad E = -\frac{109}{36}\right)$$

where E = Y-co-ordinate of vertex. The y-co-ordinate is $e = 3E$ (since Y-unit $= 3 \times y$-unit).

$$\therefore\quad e = -\frac{109}{12} = -9\tfrac{1}{12}$$

Thus position of vertex in x, y-co-ordinates is: $\left(\frac{5}{6}, \; -9\frac{1}{12}\right)$.

Place vertex of stencil at this point and draw the graph. See Chart 63.

With a little practice you will find this method quicker and more accurate than plotting a set of points.

Ex. 10.31. Plotting General Quadratics

Plot the following quadratics by changing scale and origin and using the stencil.

1. $5x^2 - 3x - 2$
2. $2x^2 + 7x + 3$
3. $2x^2 + 9x - 5$
4. $4x^2 - 10x + 4$
5. $10x^2 - 9x + 2$

Solution of Quadratic Equations

You have now learnt two ways of plotting a quadratic. One way is to make a table as on p. 207. The other is to use the parabolic stencil. The rules for its use are:

1. Calculate Y, d, e.
2. Mark out x- and Y-scales.
3. Draw parabola with stencil; and insert in the proper position, i.e. with the vertex at $x = d, y = e$.

Of what use is the stencil method? In Chapter 9 we used graphs for solving

equations. We can do the same here. We learnt in Book I how to solve quadratic equations algebraically, if the equation is:

$$0 = ax^2 + bx + c$$

The formula for the solution is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula method is the most accurate and probably the quicker, but the graphical method is useful and important. With higher equations it may be all-important. It can be carried out for quadratics either with or without the stencil.

In Chapter 9 we saw that the solution of an equation amounts to finding the intersection of the graph of the equation with the line $y = 0$, i.e. the x-axis. We notice at once that there must be two solutions since the parabola cuts the x-axis twice. For example, in Chart 63 the points of intersection are approximately $2\cdot6$ and $-0\cdot9$. These are therefore the solutions—or *roots* as we call them—of the quadratic $0 = 3x^2 - 5x - 7$.

It is always useful to be able to check results. We can use the formula to check the graphical result or vice versa. Thus:

$$x = \frac{+5 \pm \sqrt{25 + 4 \cdot 7 \cdot 3}}{2 \cdot 3} = \frac{5 \pm \sqrt{109}}{6}$$

$$= \frac{5 + 10\cdot44}{6} = 2\cdot57$$

or

$$\frac{5 - 10\cdot44}{6} = -0\cdot91 \text{ (approximately).}$$

We can read off the solutions for our family of parabolas in Chart 62. Graphs A, B, C do not cut the x-axis at all. Corresponding to $y = 0$, these three quadratics have no roots *within our present meaning of the word.* This means we must think up some new algebraic rules if we want to find roots for them. Graphs D, E, F *touch* the x-axis at one point only. Thus:

$y = (x + 4)^2$ has a single root $x = -4$

$y = x^2$ has a single root $x = 0$

$y = (x - 4)^2$ has a single root $x = +4$

Graphs G, H, I have these roots:

$$y + 4 = (x + 4)^2 \text{ has } x = -6 \text{ and } -2$$
$$y + 4 = x^2 \qquad \text{ has } x = -2 \text{ and } +2$$
$$y + 4 = (x - 4)^2 \text{ has } x = +2 \text{ and } +6$$

Ex. 10.32. Solution of Quadratics by Stencil Method

From the graphs which you drew in Ex. 10.31 read off the roots of the equations corresponding to $y = 0$. Check by formula.

To Test for Real Roots

We have seen that $+ 49$ is equivalent either to $+ 7^2$ or to $- 7^2$. We call $+ 7$ and $- 7$ the two *real* roots of $+ 49$. A negative number such as $- 49$ has no real roots. It is useful to find out in advance whether an equation has any *real* roots. This depends on the quantity

$$\sqrt{b^2 - 4ac}$$

in the formula. If $b^2 - 4ac$ is negative we cannot express its square root by means of any numbers of the type we have used hitherto. We then say that the quadratic has no real roots, e.g.:

$$y = 4x^2 - 6x + 5$$

Here $\qquad a = 4 \qquad b = -6 \qquad c = 5$

$$\therefore \quad b^2 - 4ac = 36 - 80 = -44$$

$\sqrt{-44}$ is *imaginary*.

Thus the equation has no real roots. This is not the end of the story but the beginning of another story, and a very intriguing one which we shall tell later.

Ex. 10.33. Further Quadratics

Test the following for real roots by formula. Plot graphs by stencil. Where roots are real find them from the graph. Check by formula.

1. $y = 8x^2 - 2x + 3$
2. $y = x^2 + 5x + 7$
3. $y = 90x^2 + 33x + 34$
4. $y = 4x^2 + 20x + 15$
5. $y = 25x^2 + 10x - 8$
6. $y = x^2 - 6x + 10$
7. $y = 36x^2 + 24x - 5$
8. $y = 4x^2 + 4x - 7$
9. $y = 144x^2 - 72x - 7$
10. $y = 2x^2 - 4x + 5$

§ 4. *PLOTTING BY DIFFERENCES

In the following table based on the function $y = x^2$, the symbol Δ^1 stands for the result of subtracting the y term to the left from the y term to the right immediately above, and the symbol Δ^2 stands for the result of subtracting the corresponding Δ^1 term on the left from that of the right. Thus Δ^1 shows the steps by which y increases for equal steps ($\frac{1}{2}$) of x, and Δ^2 the steps by which Δ^1 increases simultaneously.

x	0		$\frac{1}{2}$		1		$1\frac{1}{2}$		2		$2\frac{1}{2}$
y	0		$\frac{1}{4}$		1		$2\frac{1}{4}$		4		$6\frac{1}{4}$
Δ^1		$\frac{1}{4}$		$\frac{3}{4}$		$\frac{5}{4}$		$\frac{7}{4}$		$\frac{9}{4}$	
Δ^2			$\frac{2}{4}$		$\frac{2}{4}$		$\frac{2}{4}$		$\frac{2}{4}$		

Chart 64 represents the above table. The y values are in blue. The Δ^1 values are in yellow. The Δ^2 values are shown in red. The blue y-lines go up by steadily increasing steps. These steps themselves, the yellow Δ^1 lines, as shown in the upper figure, go up by *equal* steps. The equal steps are the red Δ^2 lines.

If we are given the red Δ^2 line and any one of the yellow Δ^1 lines we can quickly find all the others. Then knowing the Δ^1 lines we can, given a blue starting-line, find all the other blue lines and so plot our parabola.

To find one red line we need to know two yellow ones. To find two yellow lines we need to know three blue ones. Therefore if we know three values of y we can obtain all the other values. This gives us a very quick way for plotting a parabola and it does not depend on altering scale or origin. And as you see it can be used for fractional values of x. We might want to plot very accurately, say, every $\frac{1}{10}$th unit. Here is the calculation:

x:	0		$\frac{1}{10}$		$\frac{2}{10}$	
x^2:	0		$\frac{1}{100}$		$\frac{4}{100}$	(blue)
Δ^1:		$\frac{1}{100}$		$\frac{3}{100}$		(yellow)
Δ^2:			$\frac{2}{100}$			(red)

We can now form the Δ^1 series:

$$\frac{1}{100} \quad \frac{3}{100} \quad \frac{5}{100} \quad \frac{7}{100} \quad \frac{9}{100} \dots$$

Starting from the 0-value for x^2 we then form the y series:

$$0 \quad \frac{1}{100} \quad \frac{4}{100} \quad \frac{9}{100} \quad \frac{16}{100} \quad \frac{25}{100} \dots$$

Although both the above examples are based on the equation $y = x^2$, we could plot any other equation by the same method, e.g.:

$$y = 3x^2 - 4x - 6$$

We *need* not start from $x = 0$. Let us start from $x = -2$, and take intervals of $\frac{1}{5}$.

$x = -2.$ $\qquad y = 3 \cdot 4 + 4 \cdot 2 - 6$

$\qquad\qquad = 12 + 8 - 6 = 14$

$x = -1\frac{4}{5} = -\frac{9}{5}$

$$y = 3 \cdot \tfrac{81}{25} + 4 \cdot \tfrac{9}{5} - 6$$
$$= \frac{243 + 180 - 150}{25} = \frac{273}{25}$$

$x = -1\frac{3}{5} = -\frac{8}{5}$

$$y = 3 \cdot \tfrac{64}{25} + 4 \cdot \tfrac{8}{5} - 6$$
$$= \frac{192 + 160 - 150}{25} = \frac{202}{25}$$

Take $\frac{1}{25}$ as unit for y and $\frac{1}{5}$ for x. We then form our difference table:

x ($\frac{1}{5}$ths)	-10		-9		-8
y ($\frac{1}{25}$ths)	350		273		202
Δ^1		-77		-71	
Δ^2			6		

Thus our Δ^1 series is:

$$-77 \qquad -71 \qquad -65 \qquad -59 \qquad -53 \ldots$$

And our y series is:

$$350 \qquad 273 \qquad 202 \qquad 137 \qquad 78 \qquad 25 \ldots$$

We then mark our x-axis out in $\frac{1}{5}$ths and our y-axis in $\frac{1}{25}$ths and plot as many points as we wish. Do this as an exercise.

If we use the graphical method of Chart 64 we do not actually need to obtain the y series of values. When we have calculated three values we can draw the x, Δ^1 graph. By use of a pair of dividers we can then transfer the Δ^1 values to the x, y graph. Thus we can plot a parabola from only three calculations, the rest of the work being purely graphical.

Generally it is best to begin with $x = 0$ and plot in both directions. To find the vertex, notice that the first differences decrease as you approach the vertex and then increase as you move away from it.

Ex. 10.41. Plotting by Differences

Draw graphs similar to Chart 64 of the following quadratics, starting with the 3 values shown. Plot 10 points for each parabola.

1. $y = 3x^2$	$x = 0,\ 1,\ 2$
2. $y = 2x^2 - 4$	$x = 0,\ \frac{1}{2},\ 1$
3. $y = 5x^2 - 2x + 1$	$x = 0,\ -\frac{1}{10},\ -\frac{2}{10}$
4. $y = 4x^2 + 6x - 10$	$x = -\frac{1}{4},\ -\frac{2}{4},\ -\frac{3}{4}$
5. $y = \frac{1}{5}x^2 - x + 1$	$x = 0,\ 2\frac{1}{2},\ 5$

Ex. 10.42. More Plotting Practice

Locate the vertex for each of the parabolas whose equations are given in Ex. 10.33. Then starting from $x = d$, find y for $x = d$, $x = d + f$, $x = d + 2f$, where f has suitable values which you can choose for yourself, whole or fractional. Your aim is to plot a fairly well-rounded parabola for each. Choose a suitable scale-ratio, generally either 5, 10 or 50. Find Δ^1, Δ^2 and the y series. Then plot.

§ 5.* GEOMETRICAL REPRESENTATION OF QUADRATICS

A histogram can show us how a discontinuous function, a graph how a continuous quadratic function *grows*. By means of a figurate, as in Book I, we can also visualize the build-up of a particular numerical value of any discontinuous quadratic function. Such numerical values being based on the number of dots in the figure are integers. If we want to represent the build-up of a particular value of a continuous quadratic function, we have to use bricks of a different sort. Rows of dots can stand for whole numbers only; but lines can stand for measurements which need not tally with an exact number of divisions on a yardstick. So we shall now visualize the *continuous* quadratic function by means of lines as in Chart 65.

The geometrical figure which represents a quadratic function which is not itself a perfect square is the *difference between two superimposed squares*. In the chart, the function is $x^2 + 10x - 39$. Now $x^2 + 10x + 25$ is a perfect square, i.e. $(x + 5)^2$. We can therefore write our function in the form:

$$x^2 + 10x + 25 - 39 - 25 = (x^2 + 10x + 25) - 64$$
$$= (x + 5)^2 - 8^2$$

This is the principle underlying the solution of the equation:

$$x^2 + 10x - 39 = 0$$

A name for the operation explained on p. 152 in Chapter 7 (Book I) is the method of solution by *completion of the square*. It dates back to the time before algebraic symbolism came into use, when all calculations had to be done by some visual method, geometrical or figurate.

Ex. 10.51

Draw diagrams like the one in Chart 65 for:

1. $x^2 + 4x - 45$.
2. $x^2 + 6x - 27$.
3. $x^2 + 10x - 56$.
4. $x^2 + 7x - 12\frac{3}{4}$.
5. $x^2 + 3x - 40$.

The two factors of the quadratic of Chart 65 ($x^2 + 10x - 39$) are $(x + 13)$ and $(x - 3)$. Thus the equation $x^2 + 10x = 39$ has two roots: $x = -13$ and $x = 3$. The chart shows us the clue to one answer. If $x^2 + 10x - 39 = 0$:

$$(x + 5)^2 - 8^2 = 0$$
$$\therefore\ (x + 5)^2 = 8^2$$
$$\therefore\ x + 5 = 8$$
$$\therefore\ x = 3$$

How can we visualize the alternate solution? We can do so, if we apply a trick on which we rely when we use a grid to plot a graph. A negative sign in front of a number gives it *direction*. Let us therefore make the same construction on the background of a grid, treating measurements *upwards* or to the *right* of the origin as positive, and measurements *downwards* or to the *left* as negative. Now the law of signs is:

$$(+) \times (+) \equiv (+) \qquad (+) \times (-) \equiv (-)$$
$$(-) \times (+) \equiv (-) \qquad (-) \times (-) \equiv (+)$$

Since the area of an oblong (square or rectangle) is the product of its length and breadth, the use of this convention means that we have to give:

(*a*) a *positive* sign to areas in the right top quadrant (*plus* × *plus*) or left bottom quadrant (*minus* × *minus*);

(*b*) a *negative* sign to areas in the left top (*plus* × *minus*) or right bottom (*minus* × *plus*) quadrant.

Chart 66 shows positive measurements (*upwards* and *rightwards*) in red, and negative measurements (*downwards* and *leftwards*) in blue. Areas bounded by two adjacent blue or two adjacent red lines are therefore positive, and areas bounded by adjacent lines of different colours are negative.

Ex. 10.52. Positive and Negative Areas

Draw rectangles as in Chart 66 to represent the following areas. Write the value of the net area over each rectangle.

1. $(+4) \times (+6)$
2. $(+3) \times (-5)$
3. $(-2) \times (+4)$
4. $(-5) \times (-5)$
5. $(+3-4) \times (-2)$
6. $(+3) \times (+5-4)$
7. $(+3-1) \times (+1-3)$
8. $(+2-2) \times (+4-4)$
9. $(+4-2) \times (+2-4)$
10. $(+3-3) \times (+3-3)$

Chart 65 is not a graph. It does not show how the function $x^2 + 10x - 39$ *grows* by exhibiting the successive values it has as x increases or gets smaller. It is merely an adaptation of the grid method to visualize negative as well as positive values of x. When the value of the quadratic is zero, x must be 3 if x is positive; but x need not be positive. The value of the quadratic is zero if $x = -13$. Chart 67 shows you how it is possible to visualize the same quadratic as the difference between two squares, when x is negative. With due regard to *signs*, the areas which represent $(x^2 + 10 + 25)$ add up to 64, when $x = 3$ and when $x = -13$.

Ex. 10.53. Geometric Representation of Quadratics

1. Take the quadratic $x^2 + 10x + 25$ and give x all the even-number values from -16 to $+10$. The values go in pairs, e.g. $x_1 = -12$ and $x_2 = +2$, both give us $(x+5)^2 = 49$. Draw an area diagram for each pair as in Chart 67. Then plot the parabola for the whole set of values.
2. Draw area diagrams for the double solutions of the equations in Ex. 10.51.

§ 6.* FIGURATE FUNCTIONS

This chapter began with an examination of the histogram for $Q_n = n^2$, and of the graph for the corresponding continuous function $y = x^2$. Chart 68 shows the histogram for the triangular numbers:

$$1, 3, 6, 10, 15$$

The general formula is $T_n = \frac{1}{2}n(n+1)$ and the corresponding continuous quadratic function is:

$$\begin{aligned} y &= \tfrac{1}{2}x(x+1) \\ &= \tfrac{1}{2}x^2 + \tfrac{1}{2}x + 0. \end{aligned}$$

This is of the general type $y = ax^2 + bx + c$, in which $a = \frac{1}{2} = b$ and $c = 0$. Chart 68 shows that the limiting line bounding the histogram T_n is a parabola.

Ex. 10.61. The Triangular Function

1. Draw the histogram for T_n for both positive and negative values of n, first using equal scales for x and y, then using a y scale whose unit is $\frac{1}{10}$ s.p.u. In the latter plot as far as $n = \pm 15$.
2. Draw graphs for the values in question 1, using the same scales.
3. Read off the position of the vertex.
4. What are the two solutions of the equation $y = \frac{1}{2}x^2 + \frac{1}{2}x$?
5. Draw area diagrams for the values T_{-3} to T_3.

Ex. 10.62. Other Figurate Histograms and Graphs

This is a rather elaborate revision exercise. Take the general formula:

$$F_n = 1 + (s - 1)(n - 1) + (s - 2)T_{n-2}$$

This is the formula for plane figurate numbers, with s sides and rank n, which we studied in Book I. Give s the values 3, 4, 5, 6 and you will obtain the formulae for the triangular, square, pentagonal and hexagonal numbers of type A.

1. Calculate the values of each for $n = 1, 2, 3$. Make a difference table and so continue the series up to $n = 8$ and back to $n = -8$.
2. Draw the four histograms for these series, taking 2 s.p.u. for your n unit and $\frac{1}{4}$ s.p.u. for your F unit.
3. Now express these as continuous functions, replacing n by x and F by y. They are all quadratics. Use the graphical method of Chart 60 to plot the parabolas for these functions.
4. Draw area diagrams for each quadratic for even values of x from -4 to $+4$.
5. Now repeat the whole process for the central figurate numbers whose general formula is:

$${}_sM_n = 1 + s \,.\, T_{n-1}$$

§ 7. MIXED SIMULTANEOUS EQUATIONS

You learnt in the last chapter how to solve two linear equations *simultaneously*, i.e. how to find a value for (x, y) which satisfied both equations. The same general method applies if one or other or both of the equations is a quadratic.

The roots of the two equations are the co-ordinates of the points where their graphs intersect.

We shall consider a simple case first: i.e. how to solve simultaneously the two equations:

$$(1)\ y = x^2 \quad \text{and} \quad (2)\ y = x$$

(1) is a quadratic and (2) is a linear equation. What are the values of x and y which satisfy both equations? We draw the two graphs (Chart 69). They intersect at two points:

$$(x = 0, y = 0) \quad \text{and} \quad (x = +1, y = +1)$$

These values satisfy both equations. They are therefore the simultaneous roots.

You can learn quite a lot from this single example. If you put a straight edge across the parabola in Chart 69 in as many positions as you like, you will see that a straight line cuts a parabola in not more than two points. It may just graze it at one point (we can regard this as the two points running together) or it may miss it altogether. The co-ordinates of the point of intersection are called the *real roots*. Thus there are 2, 1 or 0 real roots.

Further, we know that we can reduce *any* quadratic to the form $y - e = (x - d)^2$ and therefore Chart 69 is typical of the simultaneous solution of any linear and any quadratic equation. As usual we make a numerical check of our solution.

1st roots: (0, 0). Substituting in

$$(1)\ y = x^2, \text{ gives: } 0 = 0^2. \quad \text{Correct.}$$
$$(2)\ y = x, \text{ gives: } 0 = 0. \quad \text{Correct.}$$

2nd roots: (1, 1). Substituting in

$$(1)\ y = x^2, \text{ gives: } 1 = 1^2. \quad \text{Correct.}$$
$$(2)\ y = x, \text{ gives: } 1 = 1. \quad \text{Correct.}$$

Ex. 10.71. Simultaneous Graphical Solutions

Obtain the simultaneous solutions of $y = x^2$ with the following linear equations:

1. $y = -x$
2. $y = 3x$
3. $y = \frac{1}{2}x$
4. $y = -\frac{1}{4}x$
5. $y = 2x - 4$
6. $y = 4x + 5$
7. $y = -6x + 3$
8. $y = -10x + 4$
9. $y = \frac{3}{4}x - \frac{4}{5}$
10. $y = -\frac{1}{8}x - \frac{2}{3}$

Square Roots by Graph

We can write the equation $y = x^2$ in this form:

$$x = \sqrt{y}$$

This gives us a useful application of the above method. We can find the solution of the equation:

$$x = \sqrt{y}, \text{ when } y = 1, 2, 3 \ldots$$

From graphs, we can thus make up a *square root table.* To do this we simply place a straight edge horizontally across the parabola cutting the y-axis at 1, 2, 3 . . . and read off the x-co-ordinates of the points where these lines cut the parabola. In each case, the two roots are equal but opposite in sign. We need not confine ourselves to whole numbers. If we have chosen a big enough scale we can find the square roots for quarters or halves or tenths or any other interval.

Ex. 10.72. Square Roots

By means of the stencil draw a parabola for values of x from -5 to $+5$. Read off the square roots of all whole numbers from 0 to 25, to one decimal place. Enter in a table.

The General Quadratic

To solve simultaneously a linear equation and a quadratic, we can either plot the quadratic graph by the Difference Method or locate its vertex, change the y-scale and use the stencil. In either case *we must plot the linear equation on the same scale as the quadratic.* If you are inventive, it may also have occurred to you that you could draw your parabola once and for all on tracing-paper or cellophane and simply slide it about on your graph-paper to any required position. On top of this you can move a straight line, also drawn on tracing-paper, and so rapidly read off the solution of any pair of equations, one linear and one quadratic, without any further drawing. There is only one complication. You must read the y-scale differently in each case. So long as you are careful about this, you should have no difficulty. One practical hint: when you have traced your parabola draw a horizontal line fairly high up from one side of the parabola to the other. Join the mid-point by a straight line to the vertex. This is the *axis* of the parabola and must always lie parallel to the y-axis of the graph-paper. This helps you to fix the direction accurately. Of course, a parabola can be tilted at any angle; but *in this chapter* we have dealt only with the upright kind. Later,

you will see that a universal y-scale can be devised to overcome the scale-difficulty. In the next exercise use whichever of these three methods suits you best.

Ex. 10.73. Simultaneous Solution of the General Quadratic and Linear Equations

Solve the following pairs of equations:

1. $y = x^2 + 2$
   $y = x + 4$
2. $y = x^2 - 4x + 5$
   $y = -x + 9$
3. $y = 2x^2$
   $y = 2x + 12$
4. $y = x^2 + 10x + 30$
   $4y = -8x - 21$
5. $y = 4x^2 + 16x - 4$
   $y = 4x - 4$
6. $y = 10x^2 - 20x - 90$
   $y = 40x - 150$
7. $y = 6x^2 - 12x - 54$
   $y = -6x - 6$
8. $3y = x^2 - 8x + 13$
   $3y = -2x + 8$
9. $y = 3x^2 + 30x + 90$
   $2y = -3x + 18$
10. $10y = x^2 + 4x - 1$
    $40y = -x - 28$

CHAPTER 11

The Growth of Solids

§ 1. FACTORS AND RECTANGLES

In the last chapter, and in Book I, Chapter 7, we have learned to recognize Quadratics as:

(*a*) equations having two roots,
(*b*) formulae for series of plane figurate numbers,
(*c*) equations of parabolas,
(*d*) functions whose second differences are constant,
(*e*) formulae for the difference between the areas of two squares.

We could now, if we liked, go on studying further properties of the parabola and similar curves; but it is generally more interesting to break fresh, than to stay digging further into familiar ground. So our next step will be a step into the third dimension. We shall go on to study the family of curves based on the *cube*. We call these functions *cubics*.

Like quadratics, cubic functions may be discontinuous or continuous. We have already met the former in Book I. The patterns for cubics are 3-dimensional figurates, the formulae for which involve three factors each containing the first power of n. We may write the general expression as:

$$F_n = p(n + a)(n + b)(n + c)$$

If $p = 1$ and $a = 0 = b = c$ we have the *cubes* of Chart 5. If $p = \frac{1}{6}$ and $a = 1$, $b = 2$, $c = 3$ we have the *tetrahedral* numbers which tell us how many cannon-balls there are in a pyramid made up of them, if there are n layers of them and n of them in each of the three edges of the base. If we multiply the above factors our expression becomes:

$$F_n = pn^3 + p(a + b + c)n^2 + p(ab + ac + bc)n + pabc$$

Since p, a, b, c are fixed numbers for any particular cubic, we can replace any expression involving only such numbers by other constants, e.g. $q = p(a + b + c)$, $r = p(ab + ac + bc)$ and $s = pabc$. Our general formula then becomes:

$$F_n = pn^3 + qn^2 + rn + s$$

Hence also:

$$F_{n+1} = p(n+1)^3 + q(n+1)^2 + r(n+1) + s$$
$$= pn^3 + (3p+q)n^2 + (3p+2q+r)n + (p+q+r+s)$$
$$\therefore \quad F_{n+1} - F_n = \Delta F_n = 3pn^2 + (3p+2q)n + (p+q+r)$$

If we replace the constants in the above by $k = 3p$, $l = (3p+2q)$ and $(p+q+r) = m$, we have

$$\Delta F_n = kn^2 + ln + m$$

Thus the first difference $\Delta F_n (= F_{n+1} - F_n)$ is a quadratic. Hence the second difference $(\Delta^2 F_n)$ is the first difference of a quadratic and is therefore a linear function. The third difference $(\Delta^3 F_n)$ is therefore constant, and the fourth vanishes.

For example, here is the difference table for the *tetrahedral* numbers (Chapter 20, Book IV) of Chart 71 showing their general formula: $\frac{1}{6}(n^3 + 3n^2 + 2n)$:

$$\begin{array}{lcccccccccccc}
{}^3_3F_n & 0 & & 1 & & 4 & & 10 & & 20 & & 35 & \ldots \\
\Delta^1 & & 1 & & 3 & & 6 & & 10 & & 15 & & \ldots \\
\Delta^2 & & & 2 & & 3 & & 4 & & 5 & & & \\
\Delta^3 & & & & 1 & & 1 & & 1 & & & & \\
\Delta^4 & & & & & 0 & & 0 & & & & &
\end{array}$$

You will notice that the numbers of the 2nd line (*first* differences) are triangular numbers which are themselves quadratic functions, those of the 3rd line (*second* difference) being the natural numbers which are an A.P. We may now make the following comparison between linear, quadratic and cubic functions:

Linear	*Quadratic*	*Cubic*
1. The general formula contains no power higher than:		
The *first* power of the variable n	The *second* power of the variable n	The *third* power of the variable n
2. The general formula contains no more than:		
2 *constants*	3 *constants*	4 *constants*
3. The figurate pattern is:		
1-dimensional (*line*) or a 2-dimensional *hollow* plane figure	2-dimensional	3-dimensional
4. The *first* differences are:		
Constant	A *linear* series	A *quadratic* series

5. The *second* differences are:

zero	*constant*	*a linear series*

6. The *third* differences are:

zero	*zero*	*constant*

7. The fourth differences are:

zero	*zero*	*zero*

If we represent the law of growth of a solid number by a histogram as in Charts 70–71, we can add a new item to our table 1. For *large* values of n:

8. The tops of the columns may lie *above* or *below* the base line	The tops of the columns *all* lie *above* the base line	The tops of the columns may be *above* or *below* the base line

In other words, a cubic like a linear function extends indefinitely into the domain of *negative* as well as positive numbers. A quadratic *may* intrude into the negative domain. If so, it turns back in its course and extends indefinitely into the positive domain in both directions.

Ex. 11.11

1. Make a table of n^3 from $n = -5$ to $n = +5$ on graph paper taking 1 s.p.u. on the vertical scale as 10.
2. Draw the smooth curve passing through the mid-points of the extremities of the columns.
3. Make histograms like the above for:
   (*a*) the sum of the first n triangular numbers.
   (*b*) the sum of the first n squares.
4. Write the formulae for the above in the form $pn^3 + qn^2 + rn + s$.
5. What are the values of the constants p, q, r, s in 3(*a*)?
6. Make a difference table for n^3 and 3(*a*).
7. Make histograms for each of the rows of the difference tables for n^3 and 3(*a*).
8. Make a *cubic stencil* for the elementary cubic $y = x^3$ shown in Chart 70 as follows:
   1. Take 2 s.p.u. for x-unit and let 1 s.p.u. = 100 y-units.
   2. Calculate x^3 for all values from $x = 0$ to ± 10.
   3. Plot the graph.

4. Paste on cardboard and cut out smoothly with a razor-blade, dividing the cardboard into two curved halves. You can use either half as your stencil.

*　　*　　*　　*　　*

The graphs of Ex. 2 and 8 above are of a particular cubic for which the constants have the values $p = 1$, and $q = 0 = r = s$. Let us now make a table of values for a more representative cubic of which q, r and s have different values, e.g., as in Chart 72:

$$y = x^3 - 6x^2 - 24x + 64$$

We make a table as follows:

$x =$	-6	-4	-2	0	$+2$	$+4$	$+6$	$+8$	$+10$
$x^3 =$	-216	-64	-8	0	$+8$	$+64$	$+216$	$+512$	$+1000$
$-6x^2 =$	-216	-96	-24	0	-24	-96	-216	-384	-600
$-24x =$	$+144$	$+96$	$+48$	0	-48	-96	-144	-192	-240
$+64 =$	$+64$	$+64$	$+64$	$+64$	$+64$	$+64$	$+64$	$+64$	$+64$
Total	-224	0	$+80$	$+64$	0	-64	-80	0	$+224$

Plot this on squared paper and compare your graph with that of Chart 72. Notice the curve cuts the x-axis at 3 places which means (as the table also shows) that $y = 0$ for 3 different values (-4, $+2$, $+8$) of x. From what we have already learnt about quadratics we can draw a new conclusion. There *can* be 3 separate answers ("roots") for a cubic equation of the type:

$$px^3 + qx^2 + rx + s = 0$$

We can use a graph for finding each of these separate answers, as we use a graph to find the 2 roots of a quadratic. The principle is the same. Put

$$y = px^3 + qx^2 + rx + s$$

Plot the graph of y and find the values of x for which $y = 0$.

Ex. 11.12

Solve graphically the following:

1. $x^3 + 6x^2 + 11x + 6 = 0$
2. $x^3 - 7x = 6$
3. $x^3 - 3x^2 = -4$
4. $x^3 + 2x^2 = x + 2$
5. $6x^3 + x^2 - 5x = 2$

§ 2.* THE BUILD-UP OF A CUBIC FUNCTION

We have seen that a cubic equation can have 3 roots. Can it have more? To answer this, it is necessary to examine the build-up of a cubic function from a different angle. The histogram, the figurate pattern and the graph are three of four different ways in which we have learned to visualize a quadratic. In Chapter 10 we studied the quadratic as an *incomplete square*. In Book I, Chapter 7, we studied the algebraic solution of quadratics and one method we used was to solve them by *factors*. This also lends itself to a simple geometrical interpretation, and one which will be helpful in studying cubic and other equations.

Now bear in mind that y means a function which passes through new values as x changes. The parabola represents all possible, *whole* or *fractional*, values. It cuts the x-axis at two points. We call the x-values of these two points the *roots of the equation.* The y-values of these two points are both, of course, zero, since they are both on the x-axis, whose equation is: $y = 0$. Now the important thing about the above factorization is that the *factors tell us the roots.*

Take the quadratic:

$$y = x^2 + 2x - 24$$

We can factorize this as follows:

$$y = (x - 4)(x + 6)$$

Put $x = 4$

$$y = (4 - 4)(4 + 6) = 0 \,.\, (4 + 6) = 0$$

$\therefore \quad x = 4$ is one root.

Put $x = -6$

$$y = (-6 - 4)(-6 + 6) = (-6 - 4) \,.\, 0 = 0$$

$\therefore \quad x = -6$ is the other root.

In general, if g is one root and h the other, we can factorize the quadratic thus: $y = (x - g)(x - h)$. Now we saw in Book I (Chapter 5) that any product of two factors can be represented as a rectangle. We were then dealing with whole numbers only; and our rectangles were *dot patterns.* In this Book we deal with continuous numbers and represent them by *lines.* Thus we can represent our quadratic by a rectangle; but of course we can represent only a particular value in this way. This method is similar to the incomplete square method and has the same convention for $+$ and $-$. Of course, the total area must be the same; but the shape is a rectangle directly representing the two factors, instead of a square. Rectangular numbers will play an important part in helping us to

understand the whole figurate family of Book IV. They can be equally useful in studying the more general functions of this Book.

In Chart 74 we have plotted the graph of the function:

$$y = (x - 4)(x + 6)$$

On the left we show five values of y represented as rectangles. By our convention, positive areas are outlined in red-red or blue-blue and negative areas in red-blue. The points so represented are indicated by little squares on the graph. The resultant area of each rectangle is the algebraic sum of the positive and negative areas into which the rectangle is divided. We could see the whole thing far better by means of a film. As x changed from -8 to -1 you would then see the whole figure shrinking, but the -24 (red-blue) rectangle would not change. As x passed through 0 the x^2 (blue) square would shrivel to a point. Then it would start expanding as a red square on the opposite side of the origin. The other rectangles (except -24) would likewise change colour and cross over after shrivelling to a line. It is as if we cut the whole rectangle into unit squares and put all the *plus* ones end to end, and all the *minus* ones end to end, forming two columns of little squares. Remember (see Chart 73) that y now represents the *total area* corresponding to the difference between the heights of the two columns.

Now
$$\begin{aligned} y &= (x - 4)(x + 6) \\ &= x^2 - 4x + 6x - 24 \end{aligned}$$

And if
$$x = -8:$$
$$\begin{aligned} y &= (64 + 32) - (48 + 24) \\ &= \quad 96 \quad - \quad 72 \\ &= 24 \end{aligned}$$

If $x = -6$
$$\begin{aligned} y &= (36 + 24) - (36 + 24) \\ &= 0 \end{aligned}$$

The second trial gives us one of the roots of the equation.

To sum up: we can represent the value of a quadratic by the area of a rectangle whose sides represent the factors of the quadratic for particular values of x. When the value is zero the total area of the rectangle is zero. This gives one of the roots of the quadratic. There is, as a rule, a second value of x, which makes $y = 0$. In this example it is $x = +4$.

This is not a new way of *solving* quadratics. It is merely a new way of *visualizing them*. The area of a rectangle is a *product* of length by breadth. The value of a quadratic is the *product* of two factors. It is thus natural to represent these two

factors as the length and breadth of a rectangle. From this we can go on to the *solid* rectangular *block* which is the product of

$$\text{length} \times \text{breadth} \times \text{height}$$

i.e. three factors. This gives us a picture model for the cubic equations. We can speak of super-solid rectangles for equations of higher degree; but we cannot draw these because our real space is limited to three dimensions.

Ex. 11.21. Quadratics as Rectangles

Draw rectangles to represent the following quadratics for each of the 5 x-values shown.

1. $y = (x+2)(x-1)$. $\quad x = -3, -2, -1, 0, +1$.
2. $y = (x+3)(x+1)$. $\quad x = -5, -3, -1, 0, +1$.
3. $y = (x-4)(x-2)$. $\quad x = 0, 2, 4, 6, 8$.
4. $y = (x-5)(x-3)$. $\quad x = 0, 1, 3, 5, 7$.
5. $y = (x+2)(x-2)$. $\quad x = -4, -2, 0, 2, 4$.
6. $y = (x-3)(x-3)$. $\quad x = -1, 0, 1, 3, 5$.
7. $y = (x-1)(x-5)$. $\quad x = -3, -1, 1, 3, 5$.
8. $y = (x+10)(x-10)$. $\quad x = -20, -10, 0, 10, 20$.

Ex. 11.22. Quadratics by Factors

Factorize the following quadratics. Plot their graphs. Show that the roots correspond with the factors. Draw rectangles for the root-values of x and so show that the algebraic areas of these rectangles are all zero.

1. $y = x^2 - x - 12$.
2. $y = x^2 + 5x - 6$.
3. $y = x^2 - x - 20$.
4. $y = x^2 + 9x - 10$.
5. $y = x^2 - 9x + 20$.
6. $y = x^2 - 4x + 4$.
7. $y = x^2 + 4x - 32$.
8. $y = x^2 - 2x - 80$.
9. $y = x^2 - 1$.
10. $y = x^2 + 8x - 48$.

* * * * *

The simplest cubic equation is:

$$y = x^3$$

We can tabulate a series of values for this function thus:

x:	-3	-2	-1	0	1	2	3
y:	-27	-8	-1	0	1	8	27

Note that the sign of x^3 is the same as the sign of x, e.g. $x = -2$, $x^3 = (-2) \times (-2) \times (-2) = (+4) \times (-2) = -8$. *Any* product having an *odd* number of minus-factors will be minus. We can represent each value geometrically as a cube, i.e. a rectangular solid whose length, breadth and height are all equal. We can thus use the method of representation in Chart 74, but must add a third dimension. Thus we have three x-axes all at right angles to each other. Each pair marks out a plane. We therefore have three planes and as you can see from Chart 75 this divides the space up into eight parts. We can call each part an *octant.* The sign of each octant is + or − according as the number of minuses is 0 or 2 (making +) or 1 or 3 (making −). To help you to identify each octant and its sign the chart shows you eight cubes each labelled according as it is Right (R) or Left (L), Over (O) or Under (U), and Front (F) or Back (B). Study the chart carefully and you will see that four are all *positive*:

ROF, LOB, RUB, LUF

The remaining four are all *negative*:

ROB, LOF, RUF, LUB

Since all these eight cubes are of the same size, half being +, half −, their net volume is zero. Suppose now we had a solid rectangular block placed in such a way that some part of it fell in each of the eight octants. If we knew the length, breadth and height of each of the eight parts we could calculate the volume of each, give it its correct sign, find the algebraic sum, and so find the net volume of the whole block. Chart 76 gives you an example. The three factors in the cubic have two terms each, thus there are 2^3 products, i.e. 8. Each product is represented by a rectangular block, whose volume is + or − according to its position. The net volume, y, is the algebraic sum of the 8 blocks. For example, the block $(ROF)_1$ is $x \cdot x \cdot x = x^3$, i.e. 1000. The block $(LOB)_2$ is $(-8)(-2)(+4) = +64$. The block $(LOB)_1$ is out of sight. Study the chart carefully before doing the next exercise.

Ex. 11.23. CUBIC CALCULATIONS

1. Copy Chart 76 but leave out the blocks $(ROF)_1$ and $(ROF)_2$.
2. Draw $(LOB)_1$ by itself in its correct position on the three axes.
3. Name all the + blocks and calculate their total volume.
4. Name all the − blocks and calculate their total volume.
5. Verify that the difference of the last two answers is 224.

6. Multiply out the factors

$$(x-2)(x+4)(x-8)$$

by finding the product of the first two, then multiplying by the third. Substitute $x = 10$ in the resulting expression and verify that the value of the expression is 224.

7. Note that three different quadratics can be formed from the above expression by taking two of the factors only. Each can be represented by a plane rectangle as in § 1. Draw these three plane rectangles. Compare them with those of Chart 76. What do you notice?

8. Draw the block diagram of this same cubic function for the value $x = 5$. Find y.

9. Draw the block diagram for $(x+2)(x-4)(x+8)$ for the value $x = 3$. Find y.

10. Draw the block diagram for $(x+3)(x+4)(x+5)$ for the value $x = -4$. Find y.

* * * * *

From this treatment of the cubic we get a new slant on its build up. Just as the quadratic has *two* factors which may but need not be identical, the cubic has three factors each of which like the two factors of a quadratic is a linear function of x. Two or three of them may be identical, but they may all be different. Any cubic equation may be arranged, so that we have on one side zero and on the other a cubic function. The latter will be zero, if any one of its factors is zero. So the solution of a cubic equation is the solution of three linear equations. If $y = (x-a)(x-b)(x-c)$, y can be zero, when $(x-a)$ is zero, $(x-b)$ is zero or when $(x-c)$ is zero. So the solutions of the cubic equation $(x-a)(x-b)(x-c) = 0$ are $x = a$, $x = b$ and $x = c$.

We can now regard x^3 as a product of 3 factors:

$$(x-0)(x-0)(x-0)$$

If we represent it as a rectangular block, it will simply be a cube $x \times x \times x$ in the octant ROF for $+$ values of x and LUB for $-$ values of x. The graph represents the volume of this cube shrinking down to nothing, crossing to the opposite octant, and then expanding again. Regarded as an equation, the function $y = x^3$ is equal to zero for only one value of x, namely 0. In other words, the graph cuts the x-axis at the origin.

By direct multiplication you can easily satisfy yourself that the factors of

$x^3 - 6x^2 - 24x + 64$ are $(x-2)$, $(x+4)$ and $(x-8)$. The following table shows why $x = 2$, $x = -4$ and $x = 8$ are the roots of $x^3 - 6x^2 - 24x + 64 = 0$:

x	-6	-4	-2	0	2	4	6	8	10
$(x-2)$	-8	-6	-4	-2	0	2	4	6	8
$(x+4)$	-2	0	2	4	6	8	10	12	14
$(x-8)$	-14	-12	-10	-8	-6	-4	-2	0	2
product, y:	-224	0	80	64	0	-64	-80	0	224

*Plotting the Cubic by Graphic Differences

Chart 77 is drawn by the same Difference Method which we used for the parabola in Chapter 10. The 3rd differences are constant. Thus we need to calculate directly only four values of y, form the 1st, 2nd and 3rd differences and then by repeated additions of Δ^3 build up the values of Δ^2, Δ^1 and y. Alternatively, we can construct a straight-line graph for Δ^2 and then starting with any value of Δ^1 keep adding successive values of Δ^2 by means of a pair of dividers. This builds up the values of Δ^1, which lie, as Chart 77 shows you, on a parabola. We now repeat the process. Starting with one of the values of y already calculated go on adding Δ^1 repeatedly (with due regard for signs), by means of dividers, so building up the cubic curve. The scale presents difficulties because cubics rise to such high figures. Consequently we have here reduced the scale of the cubic to one-eighth of the scale of the parabola. It is useful to have a quick graphical method for changing scale. The footnote below * shows you how to magnify or diminish the length of a line in any ratio without recourse to calculation. It is not worth using it when you have only a single line to change, but when you have a whole series of lines (as in changing the scale for a graph) it saves valuable time. In Chart 77, the change of scale affects the y-axis only, and enables us to draw the graph within a reasonable space. Before starting the next exercise note these points:

1. The cubic cuts the x-axis at 3 points. These give the roots of the cubic, viz. 8, 2, -4.
2. The curve has a sort of hillock and a valley. We call the hillock a *maximum* and the valley a *minimum*.
3. The positions of the maximum and minimum have Δ^1 values $= 0$, where the parabola cuts the x-axis.

* Draw a triangle ABC with base AB $= 8$ in., $\angle$ABC $= 90°$, height BC $= 1$ in. To reduce any value, e.g., 3 in. to the $\frac{1}{8}$ scale, mark a point D on AB such that AD $= 3$ in. If the perpendicular from D on AB cuts AC at E, DE $= \frac{3}{8}$ in. and is the value of AD on the $\frac{1}{8}$ scale. For any other ratio $\frac{1}{n}$, make AB n inches.

4. The left-hand part of the cubic is curved downwards. The right-hand part curves upwards. Between the two is a short straight region (at $x = 2$). This is the point at which the parabola just passes *its* minimum. If you were tobogganing down the cubic you would find this point the steepest region between the hill and the valley. We call this a point of *flexion*. Thus a point of flexion corresponds with a minimum for Δ^1.
5. Note that the Δ^2 line cuts the x-axis at the point for which Δ^1 is a minimum, i.e. when Δ^1 is a minimum $\Delta^2 = 0$. (Compare this with point No. 3.)

Drawing graphs is something of an art. On the one hand, we have to trace as smooth a curve as possible. This means plotting a good many points, particularly in those regions where the graph twists about most. On the other hand, we want to avoid awkward numbers, particularly fractions if we can. Only experience can tell us the best points to plot and the most suitable scales to choose.

We need not be afraid of *large* numbers. We simply reduce our scale to take them. It is the fractions which cause difficulty. If the equation has small coefficients and we want to plot a good many points, we cannot avoid fractional values; but what we can do is to choose a fractional unit and multiply through by the denominator. We can do this to the equation we have been studying:

$$y = x^3 - 6x^2 - 24x + 64$$

If we wish to plot $\frac{1}{2}$ units of x all the x-values are now doubled. Then put $x = \frac{X}{2}$ (i.e. X means x reckoned in $\frac{1}{2}$ units).

$$y = \frac{X^3}{8} - \frac{6X^2}{4} - \frac{24X}{2} + 64$$

Multiply through by 8 :

$$8y = X^3 - 12X^2 - 96X + 512$$

Now adopt a new y-unit. It will have to be $\frac{1}{8}$th of the previous y-unit. Thus $y = \frac{Y}{8}$ $\therefore$ $8y = Y$. Our equation now becomes:

$$Y = X^3 - 12X^2 - 96X + 512$$

The coefficients are now sufficiently large to allow us to work throughout with whole numbers. We plot our graph for X, Y and then change the scale back after plotting, by halving all the figures on the X-axis and dividing all the Y-figures by 8. The latter is what we have done in plotting Chart 77. In the

next exercise we have an equation with large coefficients and there is no need to use fractional values.

Ex. 11.24

1. For the equation
$$y = x^3 + 10x^2 - 25x - 250$$
calculate the values of y for the x-values 10, 9, 8, 7, 6.

2. Find Δ^1, Δ^2 and Δ^3 for these five values of y.

3. You know that Δ^3 is constant, Δ^2 is a linear function and Δ^1 is a parabola. You have 3 values of Δ^2 and can therefore plot the straight-line graph. Take 1 s.p.u. for your x-scale unit, and let 1 s.p.u. = 10 on the y-scale. To check the accuracy of your graph note that it should rise by steps equal to Δ^3 for every unit along the x-axis.

4. Your Δ^1 values are roughly 5 times as large as your Δ^2 values. Thus a smaller scale must be used. Take 1 s.p.u. = 50 y-units. Plot the first value of Δ^2 which you have calculated for $x = 10$. Now place your dividers at the point $x = 10$ on the Δ^2 graph and measure the height of the graph at that point. Refer to the footnote on page 232, reduce this to one-fifth and apply the resulting length to the Δ^1 value you have just plotted. As the values are going *down* ($x = 10, 9, 8, 7 \ldots$) you must *subtract* this length. This gives you your second Δ^1 value. You can check it against the calculated value. Then measure Δ^2 for $x = 9$, reduce to $\frac{1}{5}$th and subtract it from your second Δ^1 line. You can also check this value. Once you are sure you have made a correct start you can continue this process as far as $x = -15$, thus building up your parabola of Δ^1 values.

5. You can now construct the cubic. Your y values on the cubic are roughly 3 times as large as the y values on the parabola, so you must once again reduce the scale, but as $\frac{1}{3}$ is awkward take $\frac{1}{2}$ as the factor. You will need a separate sheet of graph paper. Halve all the Δ^1 values by the "lens." Start by plotting the first value of $y = x^3 + 10x^2 - 25x - 250$ which you calculated for $x = 10$. Subtract the first value of Δ^1 (halved) with your dividers. Check against the calculated result for the second value (for $x = 9$). Continue in this way building up the cubic from the parabola.

* * * * *

The last exercise probably convinced you that graphical plotting by differences is both slow and inaccurate. Each point on the cubic is obtained after four

adjustments of the dividers. Thus we cannot regard this as a practical method. It was intended simply to show you that the first differences of the cubic lie on a parabola and the second differences lie on a straight line. Thus, adding successive ordinates of the straight line gives a parabola, and adding successive ordinates of the parabola gives a cubic. This helps you to understand the structure of the cubic. We can use the Difference Method accurately by building up a complete table of Δ^2, Δ^1 and y values, by repeated subtraction or addition and then plotting y direct. To plot 25 values in this way requires 75 additions or subtractions. Again, this is a slow process and a single mistake makes all subsequent results incorrect.

Ex. 11.25. Plotting the Cubic by Arithmetical Differences

Work with the same equation as in Ex. 11.24, viz.:

$$y = x^3 + 10x^2 - 25x - 250$$

Find, by substitution, the values of y for $x = -15, -14, -13, -12$.

Then mark out two pages into 5 columns each as follows:

x	y	Δ^1	Δ^2	Δ^3

Under x enter all the whole numbers from -15 to $+10$, on alternate lines.

Enter the four values of y which you have just calculated.

Find the corresponding values of Δ^1, Δ^2, Δ^3 by subtraction.

Δ^3 is constant. Start adding it repeatedly to Δ^2 (taking account of signs), entering values on alternate lines until the whole column is filled.

Now add each value of Δ^2 to its corresponding Δ^1 until the whole Δ^1 column is filled.

Finally, add each Δ^1 value to its corresponding y value so completing the table.

Now choose a suitable scale and plot the x, y graph.

The Cubic as a Sum of Three Terms

Still sticking to our equation

$$y = x^3 - 6x^2 - 24x + 64$$

we see that y can be expressed as a sum of three terms:

$$(x^3) \text{ and } (-6x^2) \text{ and } (-24x + 64)$$

Chart 72 shows these three terms plotted as separate graphs and then added together, to form one composite graph. To add graphs together we take a particular value of x, e.g. -8, measure the height of each graph (i.e. the y value)

at this point, add the heights together and plot this value on the new graph. The adding can be done graphically, using dividers. Thus:

(1)	$y = -24x + 64$	$x = -8: y = 256$
(2)	$y = -6x^2$	$x = -8: y = -384$
(3)	$y = x^3$	$x = -8: y = -512$
(4)	$y = x^3 - 6x^2 - 24x + 64$	$x = -8: y = -640$

Ex. 11.26. Graphical Addition

1. Measure the heights of each of the red dots from the x-axis in Chart 72. Add each set of 3 together and compare the sum with the height of the corresponding red dot on the 4th graph.
2. Make a similar chart for the equation $y = x^3 + 10x^2 - 25x - 250$.

§ 3. REAL AND IMAGINARY ROOTS

To solve a quadratic of the form $ax^2 + bx = c$ by means of the formula (p. 212) or by factors (Book I), we reduce it to the form $x^2 + px + q = 0$. For instance, $3x^2 + 3x = 18$ becomes:

$$3x^2 + 3x - 18 = 0$$
$$\therefore \quad x^2 + x - 6 = 0$$

We can solve this graphically if we put $y = x^2 + x - 6$ and plot the curve so defined. Our two answers, the roots of the equation, are the values of x where the curve cuts the x-axis, i.e. when $y = 0$. To plot the curve, it is not necessary to make a table of y-values corresponding to particular values of x, positive or negative. We can use our parabolic stencil (p. 207) for $y = x^2$ by the method set out on p. 208. Alternatively, we can use it in another way which is simpler. We can write the above as: $x^2 = 6 + x$. Since the expressions on the left and right are equal we can use one symbol for them, i.e. we can put:

$$x^2 = y = 6 + x$$

If we now draw the line $y = 6 + x$ on the same scale as the parabola $y = x^2$, the two answers required are the 2 x-co-ordinates of the points where the line crosses the curve.

This is the most speedy way of getting a graphical solution, or to be more precise a "real" solution. The word *real* has a particular meaning in books on algebra. We have seen that $(+7)^2 = (-7)^2 = +49$, i.e. $\sqrt{+49} = \pm 7$.

Positive numbers like $+49$ have *real* square roots, one positive, the other negative and *numerically* equal to it. A negative number such as -49 has no *real* square roots. A square root of a negative number is said to be *imaginary*. Sometimes the solution of a quadratic equation involves such imaginary roots. Here is an example:

$$2x^2 + 2x + 25 = 0 \qquad \therefore \quad x^2 + x + \frac{25}{2} = 0$$

On applying the formula of p. 212, we get:

$$x = \frac{-1 \pm \sqrt{1-50}}{2} = \frac{-1}{2} \pm \frac{\sqrt{-49}}{2}$$

Both negative and positive numbers when squared yield positive numbers. The square root of a negative number is neither a positive number which we represent to the right along the x-axis and upwards along the y-axis nor a negative number which we represent to the *left* along the x-axis and *downwards* along the y-axis. Consequently, the values of x for which a function such as $2x^2 + 2x + 25$ becomes zero do not exist *on the plane of our graph paper*. In other words, our y curve never cuts the x-axis. Alternatively, if we plot $2x^2 = y$ and $-2x - 25 = y$, there will be no points at which the line $y = -(2x + 25)$ cuts the parabola $2x^2 = y$.

§ 4. GRAPHICAL SOLUTION OF THE CUBIC EQUATION

A graphical method of solving the cubic equation is essentially like the method of solving a quadratic. We may first reduce our equation to a standard form with only 3 constants, thus:

$$2x^3 - 2x^2 - 20x = 16$$
$$\therefore \quad x^3 - x^2 - 10x = 8$$
$$\therefore \quad x^3 - x^2 - 10x - 8 = 0$$

We can now make a table of x and y values for the function

$$y = x^3 - x^2 - 10x - 8$$

If the curve cuts the x-axis at 3 places we have 3 *separate* real roots, which are the required solution. This method is not very accurate; but that is unimportant, because we can always check our solution by arithmetic. What matters more is that it is laborious. There are several ways of side-stepping the labour. If

$$x^3 - x^2 - 10x - 8 = 0$$
$$x^3 = x^2 + 10x + 8$$
$$\therefore \quad x^3 = y = x^2 + 10x + 8$$

Accordingly, one way of proceeding with the minimum of effort is to draw the graph $x^3 = y$ with a cubic stencil, and the graph $x^2 + 10x + 8 = y$ with a parabolic stencil of the same scale, as explained on pp. 209–211. Alternatively, we can plot $x^2 + 10x + 8 = y$ from a table of values prepared in the usual way.

Ex. 11.31

Find by means of a cubic and quadratic graph the *real* roots of the following equations.

1. $x^3 + 3x^2 + 12x - 16 = 0$.
2. $x^3 - 6x^2 + 30x = 63$.
3. $x^3 + 9x^2 + 33x + 52 = 0$.
4. $x^3 - 9x^2 + 18x = 28$.
5. $x^3 + 3x^2 + 4 = 3x$.
6. $x^3 + 6x^2 - 50x + 43 = 0$.
7. $x^3 + 33x = 3x^2 + 148$.
8. $x^3 + 6x^2 + 72x = -189$.

* * * * *

The graphical method just described involves less effort than solution of a cubic by tabulating y- and x- values for the function:

$$y = x^3 + ax^2 + bx + c$$

It has one disadvantage. Both the quadratic and the cubic graph *slope steeply*. Hence small inaccuracies of draughtsmanship may lead to big inaccuracies in assigning where the two curves cut each other. Let us now look at a cubic which contains no term involving x^2, e.g.:

$$x^3 - 18x - 35 = 0$$

We may write this as:

$$x^3 = 18x + 35$$

$$\therefore\ x^3 = y = 18x + 35$$

Our solution is therefore the x-values at which the straight-line graph $y = 18x + 35$ cuts the cubic $y = x^3$ drawn on the same scale. Now it is easier to draw a straight line accurately with a ruler than to fit a smooth curve to a quadratic; and even if we use the stencil method for drawing the graph of the latter the procedure involved in finding where the straight line cuts the graph of our cubic stencil is more likely to give an accurate result. It would therefore be useful to have a trick for converting a cubic of the standard form involving x^3, x^2, x and 3 constants into the form:

$$x^3 + px + q = 0$$

We have learned (p. 210) how to convert a quadratic involving x^2, x and 2 constants into the quadratic $y = x^2$. So it is worth while to explore the possibility that we can reduce a cubic to the form shown above by a similar procedure, i.e. by shifting the origin along the x-axis. Suppose that our cubic equation is:

$$x^3 + 6x^2 = 6x + 63$$
$$\therefore \quad x^3 + 6x^2 - 6x - 63 = 0$$

Let us test the possibility by *assuming* that we can produce this result by changing our origin, i.e. by putting:

$$x + d = X$$

If we can do so, we shall then get an equation of the form:

$$X^3 + pX + q = 0$$

This equation is equivalent to putting:

$$(x + d)^3 + p(x + d) + q = 0$$
$$\therefore \quad x^3 + 3dx^2 + 3d^2x + d^3 + px + pd + q = 0$$
$$\therefore \quad x^3 + 3dx^2 + (3d^2 + p)x + (d^3 + pd + q) = 0$$

This equation is equivalent to the one which we are trying to reduce to a form involving only x^3, x and 2 constants, if:

$$x^3 + 6x^2 - 6x - 63 = x^3 + 3dx^2 + (3d^2 + p)x + (d^3 + pd + q)$$

This must be so, if:

$$(a)\ 3d = 6; \quad (b)\ 3d^2 + p = -6; \quad (c)\ d^3 + pd + q = -63$$

From (a) we have:

$$d = 2$$
$$\therefore \quad 3d^2 + p = 12 + p$$

Hence from (b):

$$12 + p = -6$$
$$\therefore \quad p = -18$$

In the same way, paint in the values $p = -18$, $d = 2$ in (c):

$$\therefore \quad q = -8 + 36 - 63 = -35$$
$$\therefore \quad X^3 + pX + q = X^3 - 18X - 35 = 0$$

We can obtain a real root of this equation by finding where the straight-line graph $y = 18X + 35$ cuts the cubic $y = X^3$, which we can draw with the stencil. We thus get the real values of X. Hence we know the real values of x, since $X = x + d = x + 2$, i.e.

$$x = X - 2$$

This method of reduction is always applicable. We can generalize the argument as follows. Our complete cubic equation is:

$$x^3 + ax^2 + bx + c = 0$$

The reduced form we seek is:

$$X^3 + pX + q = 0$$

If we put $X = x + d$, we have:

$$\begin{aligned} &x^3 + 3dx^2 + (3d^2 + p)x + (d^3 + pd + q) = \\ &x^3 + ax^2 + \qquad bx \quad + \qquad\qquad c \end{aligned}$$

$$\begin{array}{ll} \therefore\ 3d = a & \therefore\ d = \dfrac{a}{3} \\ 3d^2 = \dfrac{a^2}{3} & \therefore\ 3d^2 + p = \dfrac{a^2}{3} + p = b \\ & \therefore\ p = b - \dfrac{a^2}{3} \\ d^3 = \dfrac{a^3}{27} & \therefore\ d^3 + pd = \dfrac{a^3}{27} + \dfrac{ab}{3} - \dfrac{a^3}{9} \\ \therefore\ \dfrac{9ab - 2a^3}{27} + q = c & \therefore\ q = c + \dfrac{2a^3 - 9ab}{27} \end{array}$$

If we know a, b and c, we therefore have all the information we need for finding d, p and q in the equation replacing x by $X = x + d$.

Ex. 11.32 (Chart 78)

1–8. Solve each of the equations of Ex. 11.31 by:

(a) reducing to the form $X^3 + pX + q = 0$;

(b) finding where the straight-line graph $y = -pX - q$ cuts the cubic curve $y = X^3$.

*   *   *   *   *

§ 5. ALGEBRAIC SOLUTION OF A CUBIC EQUATION

The trick we have introduced to make possible the graphical solution of a cubic by means of a straight line and a cubic stencil is the principle which underlies the algebraic method. It may happen that $q = 0$, as when (Chart 78):

$$x^3 - 6x^2 - 24x + 64 = 0$$

By applying the method of § 4 we find that this reduces to the standard form, if we put

$$X = x + d$$

We then get:

$$(x + d)^3 + p(x + d) + q = 0$$
$$\therefore \quad 3d = -6 \text{ and } d = -2$$
$$p = -36 \;:\; q = 0$$
$$\therefore \quad X^3 - 36X = 0$$

We can write this as:

$$X(X^2 - 36) = 0$$
$$\therefore \quad X(X - 6)(X + 6) = 0$$

This is true if any of the 3 factors is equal to zero, i.e. if $X = 0$ *or* $(X - 6) = 0$ or $(X + 6) = 0$. Thus there are 3 real roots:

$$X = 0;\; X = 6;\; X = -6$$

Since $X = x + d$, $x = X - d = X + 2$. Now X can have any one of the 3 values above. So there are 3 possible values of x consistent with our equation, i.e.

$$(a) \quad x = 0 + 2 = 2$$
$$(b) \quad x = 6 + 2 = 8$$
$$(c) \quad x = -6 + 2 = -4$$

The simplest way of checking this result is to build up the original equation from the roots by using factors. The cubic function is equal to zero if any one of its factors is equal to zero. If $x = 2$ one factor must be $(x - 2)$. The other 2 will be $(x - 8)$ and $(x + 4)$. By direct multiplication we have:

$$(x - 2)(x - 8)(x + 4) = x^3 - 6x^2 - 24x + 64$$

Ex. 11.41

Solve the following by the method shown above:

1. $x^3 + 9x^2 + 23x + 15 = 0.$
2. $x^3 - 6x^2 + 11x = 6.$
3. $4x^3 - 12x^2 + 3x + 5 = 0.$
4. $9x^3 - 108x^2 + 432x = 576.$
5. $x^3 - 6x^2 - 4x + 24 = 0.$
6. $4x^3 + 36x^2 + 92x = -60.$

* * * * *

Needless to say a cubic equation does not always reduce to a form so sim When q does *not* vanish we have to proceed in another way. We have seen] we can reduce to the form involving only 2 constants:

$$x^3 + 6x^2 - 6x - 63 = 0$$

When $X = (x + 2)$, this becomes:

$$X^3 - 18X - 35 = 0$$

We now introduce a new trick. Put

$$X = u + v$$
$$\therefore X^3 = u^3 + 3u^2v + 3uv^2 + v^3$$
$$= 3uv(u + v) + (u^3 + v^3)$$
$$= 3uv \,.\, X + u^3 + v^3$$

But we have above:

$$X^3 = 18X + 35$$

Hence if $X = u + v$,

$$3uv = 18 \text{ and } u^3 + v^3 = 35$$

We have now 2 equations with 2 variables. We can solve them as follows:

$$(u^3 + v^3)^2 = 35^2 = 1225; \; u^3v^3 = \left(\frac{18}{3}\right)^3 = 216$$
$$\therefore \; u^6 + 2u^3v^3 + v^6 = 1225$$
$$4u^3v^3 = 864$$
$$\therefore \; u^6 - 2u^3v^3 + v^6 = 361 = 19^2$$
$$\therefore \; u^3 - v^3 = 19$$

If we combine this with $u^3 + v^3 = 35$, we have:

$$2u^3 = 54 \quad \therefore \quad u^3 = 27 \quad \therefore \quad u = 3$$
$$2v^3 = 16 \quad \therefore \quad v^3 = 8 \quad \therefore \quad v = 2$$

Since $X = u + v$,

$$X = 2 + 3 = 5$$

Thus one root of our reduced equation is $X = 5$, and since $X - 2 = x$, one real value of x is 3, and one factor of our cubic must be $(x - 3)$. The other factor which is a quadratic is obtainable by division (see Book III, Chapter 15). In this case, as you can check by multiplication:

$$(x - 3)(x^2 + 9x + 21) = 0$$

This is true if either $(x - 3) = 0$ *or* $x^2 + 9x + 21 = 0$.

The solution of the last gives us the other two roots of our cubic. Both involve imaginary numbers, being:

$$\frac{-9 + \sqrt{-3}}{2} \quad \textit{and} \quad \frac{-9 - \sqrt{-3}}{2}$$

Ex. 11.42

Find one real root of each of the following equations:

1. $x^3 + 9x - 26 = 0.$
2. $x^3 + 3x^2 + 9x + 27 = 0.$
3. $x^3 + 6x + 7 = 0.$
4. $x^3 = 9x + 28.$
5. $x^3 + 9 = 6x.$
6. $x^3 + 36x = 37.$
7. $x^3 + 30x = 117.$
8. $x^3 + 60x + 61 = 0.$

Note.—If we can factorize a cubic (see p. 360–1, Book III) we may apply the method given for a quadratic on p. 159, Book I.

Inverse Proportion and the Hyperbola

§ 1. INVERSE PROPORTION

We have now learnt the meaning of the word *function* and of *dependence* between two variables. Graphs and histograms are visual ways of showing dependence. *Direct proportion* dealt with in Chapter 7 of Book I is a special kind of dependence. Ordinarily we mean a sort of dependence which we can summarize in the formula for an A.P. or the equal step histogram when the two variables (or one of them) grows by jumps and the linear equation or the straight-line graph when their growth is continuous. For purposes of calculation, the linear equation and the straight-line graph work either way. Here is an example:

x	1	2	3	4	5	6 . . .
y	7	11	15	19	23	27 . . .

The equation is $y = (4x + 3)$, and the straight-line graph slopes upwards from left to right. When one quantity (y) is *directly proportional* to another (x), y increases as x increases and decreases as x decreases, but this can be true of many types of dependence other than what we commonly mean by direct proportion, unless we add something more to our statement. For instance, we may say that y is directly proportional to the square of x, when y is a quadratic function of x and as such can be represented by a parabola. This is true of the following for which the equation is $y = 2x^2 + 3$:

$$x \ldots -3, \quad -2, -1, 0, +1, \quad +2, \quad +3, \quad +4 \ldots$$
$$y \ldots +21, +11, +5, 3, +5, +11, +21, +35 \ldots$$

While it is true that we can always represent direct proportion by a linear equation or straight-line graph, the converse is not true. A linear function of x may decrease, as x increases. For example, the following table shows y and x values embodied in the equation $y = -4x + 3$:

$$x \ldots -3, \ -2, -1, \quad 0, +1, +2, +3, \ +4 \ldots$$
$$y \ldots +15, +11, +7, +3, -1, -5, -9, -13 \ldots$$

Customarily, we say that y is *directly* proportional to x, only when y increases by

equal steps as x increases by equal steps or decreases by equal steps when x decreases by equal steps, i.e. when it is a linear function of x with a *positive* coefficient. Many kinds of dependence are not of this sort, including linear dependence when the coefficient of x is negative, i.e. when y *decreases* by equal steps as x increases by equal steps and *vice versa*. Another sort of dependence which exists when y increases *numerically* as x decreases *numerically* or decreases *numerically* as x increases *numerically* has a special name. We call it *inverse proportion*. Here is an example:

$$\begin{array}{lccccccc} x \ldots & -10, & -5, & -1, & 0, & +1, & +5, & +10 \ldots \\ y \ldots & -1, & -2, & -10, & ?, & +10, & +2, & +1 \ldots \end{array}$$

If we leave out of account the middle item, the peculiarity of this sort of dependence is not difficult to recognize. The *product* of corresponding values of x and y is always the same, being 10 in this example. The equation is therefore:

$$xy = 10$$

The number on the right is *fixed* for this particular function. It is a constant. One variable is always said to be inversely proportional to another, when the product of the two is a constant, i.e. if

$$xy = C \textit{ or } y = \frac{C}{x}$$

Two variables are in direct proportion, when the *quotient* is a *positive* constant, i.e. when

$$\frac{y}{x} = C \textit{ or } y = Cx$$

More generally, we may say that two variables are in direct proportion when there is a constant (and positive) ratio between one and the other increased or decreased by a fixed amount, i.e. by a second constant (here K). We may write this in the form:

$$\frac{y - K}{x} = C \textit{ or } \frac{y + K}{x} = C$$

$$\therefore \; y = Cx + K \textit{ or } y = Cx - K$$

In our first example of direct proportion (p. 244) C was 4 and K was 3, the sign of K being positive, if we use the left-hand formula. In the same way, we may speak of two variables as inversely proportional to one another, if:

$$x(y - K) = C \text{ or } x(y + K) = C$$

$$\therefore\ y = \frac{C}{x} + K \text{ or } y = \frac{C}{x} - K$$

This is true of the following:

x	$+\frac{1}{4}$,	$+\frac{1}{2}$,	$+1$,	$+2$,	$+3$,	$+4$
y	$+23$,	$+13$,	$+8$,	$+5\frac{1}{2}$,	$+4\frac{2}{3}$,	$+4\frac{1}{4}$

By testing any pair of y and x values, you can satisfy yourself that $C = 5$ and $K = 3$, i.e.

$$x(y - 3) = 5$$

We can represent dependence of this sort by a histogram when x increases by jumps, or by a graph, when x increases *continuously*. Chart 79 shows the histogram of the discontinuous function:

$$H_n = \frac{10}{n} \quad or \quad n \,.\, H_n = 10$$

Here as elsewhere, we use n in contradistinction to x to indicate that the independent variable must have *discrete* values, e.g. whole numbers. This chart shows only values of the dependent variable (H_n) for positive values of n. Why we use the symbol H will come later (p. 253). The column for H_0 is not in the chart, because it is too tall. When $n = 0$, H_n is $10 \div 0$. What does this mean? Multiplication is repeated addition. Division involves repeated subtraction. The answer tells you how many times you can go on taking the divisor away from the dividend until you have nothing left. Now you can go on taking 0 away from 10 (or any number) till you are grey-haired without finding that you have nothing left. The number of times you can do so is *indefinitely large*. We have a special sign for it, ∞, and a special name *infinity*. Infinity is not the name for any particular number. It is a label for an order of magnitude beyond our powers of describing by means of particular numbers. It is off the map.

Note also what happens when n becomes very large. H_n then becomes very small. When n is immeasurably large, so that we have to represent it by our sign for the immeasurably large numbers:

$$H_n = 10 \div \infty \quad or \quad \infty \,.\, H_n = 10$$

If we divide 10 (or any number) by a number which is immeasurably large, we get a quotient which is immeasurably small, i.e. *zero*. In general therefore:

$$\frac{n}{\infty} = 0 \quad and \quad \frac{n}{0} = \infty$$

Ex. 12.11

Draw histograms like Chart 79 for:

1. $H_n = \frac{1}{n}$
2. $n \cdot H_n = 2$
3. $H_n = \frac{4}{n}$
4. $H_n = \frac{4}{n} - 6$
5. $n(H_n - 5) = 4$

* * * * *

Chart 80 shows the graph of a function defined by the equation:

$$xy = C$$

For the present (*see* § 3 *below*) we shall assume that C is *positive*. In the chart $C = +100$. The scales for x and y are the same. If we tabulate positive values of x, we have:

$$x = \ldots \quad 0{\cdot}1 \quad 1{\cdot}0 \quad 10 \quad 100 \quad 1000 \ldots$$
$$y = \ldots \quad 1000 \quad 100 \quad 10 \quad 1 \quad 0{\cdot}1 \ldots$$

From such a table, you can see at once that:

(*a*) when x is positive all values of y are positive;
(*b*) when x gets very nearly zero, y becomes immeasurably big;
(*c*) as x gets very big, y becomes very small.

On the graph we therefore find:

(*a*) for positive values of x the curve lies wholly in the top right quadrant;
(*b*) the curve has two limits which respectively tail off very close and almost parallel to the y- and x-axes.

For x values which are negative, we can make a table such as this:

$$x = \ldots \quad -0{\cdot}1, \; -1{\cdot}0, \; -10, \; -100, \; -1000 \ldots$$
$$y = \ldots \; -1000, \; -100, \; -10, \quad -1, \quad -0{\cdot}1 \ldots$$

All y values are negative, when x is negative. Corresponding negative and positive values are numerically the same. Thus the complete graph of the function $xy = 100$ has two separate parts, one wholly in the top right quadrant, one wholly in the left bottom quadrant. The two parts never meet. One is the mirror image of the other. Each has two limbs which lie very close and almost parallel to the x- and y-axes respectively. When a limb of a curve approaches a

straight line in this way, we call it an *asymptote*, and say that it is *asymptotic* in that region to the straight line itself.

Ex. 12.12

With the same scale for x and y draw the graphs of the continuous functions corresponding to those represented by the histograms of Ex. 12.11, i.e. to

1. $y = \frac{1}{x}$
2. $xy = 2$
3. $y = \frac{4}{x}$
4. $y = \frac{4}{x} - 6$
5. $x(y - 5) = 4$

§ 2. THE HYPERBOLA

In Ex. 12.12, Nos. 1–3, the constant K is 0 and the equation is of the general form:

$$xy = C$$

The values of C are 1, 2 and 4. All three curves have much the same shape, each with two separate mirror-image parts, each part with limbs asymptotic to the x- and y-axes, and each part symmetrical about a key-point where $x = y$. The name for a curve of this sort (with its mirror-image) is the *rectangular hyperbola*. The general shape of all three rectangular hyperbolas in these exercises (Nos. 1–3, Ex. 12.12) is thus alike except in the neighbourhood of this key-point, where $x = y$, so that we can put either $x^2 = xy = C$ or $y^2 = xy = C$. At this key-point we therefore have:

$$x = \sqrt{C} \quad \textit{and} \quad y = \sqrt{C}$$

Thus if $C = 1$, the key-point in the positive quadrant is at $x = 1, y = 1$. If $C = 4$, it is at $x = 2, y = 2$. Since a square root has both positive and negative values, $x = -1 = y$ when $C = 1$, and $x = -2 = y$ when $C = 4$ in the negative quadrant. This shows us what the constant C does. It pushes the key-point, where the bend comes, further away from the origin. When C is small the bend is very sharp. When C is large the bend is less acute.

If we write the equation in the general form $x(y - K) = C$, the last 2 exercises of Ex. 12.12 contain a new constant K. What does K do? You will be able to see this more easily if you do the following exercises.

Ex. 12.21

Draw on the same scale for x and y each of the following pairs on the same grid:

1. $y = \frac{4}{x} - 6$ (Ex. 12.12, No. 4) and $y = \frac{4}{x}$ (*ditto*, No. 3).
2. $x(y - 5) = 4$ (*ditto*, No. 5) and $xy = 4$ (*ditto*, No. 3).
3. $y = \frac{100}{x} - 15$ and $y = \frac{100}{x}$ (Chart 80).
4. $y = \frac{20}{x} - 10$ and $y = \frac{20}{x}$.
5. $y = \frac{2}{x} - 4$ and $y = \frac{2}{x}$ (Ex. 12.12, No. 2).

* * * * *

When you have completed Ex. 12.21, you will have discovered that the constant K does not alter the shape of the curve at the bending point. It shifts its position on the grid, so that:

(*a*) the bending point is no longer equidistant from the x- and y-axes;
(*b*) the limb asymptotic to the x-axis approaches indefinitely near a line K y-units above it.

The second conclusion follows from the equation itself, i.e.

$$x(y - \text{K}) = \text{C} \quad \textit{or} \quad y = \frac{\text{C}}{x} + \text{K}$$

When x is indefinitely large we have:

$$y = \frac{\text{C}}{\infty} + \text{K}$$

Since $\text{C} \div \infty = 0$, $y = \text{K}$. In the region, where x is so large that $\text{C} \div x$ is negligibly small, the curve thus becomes nearly the same as the straight line whose equation (p. 191) is $y = \text{K}$. This is a line which runs parallel to the x-axis, so that the y-co-ordinate for every value of x is K. We arrive at the same conclusion, if we use a trick we have learnt elsewhere (p. 239). Put $\text{Y} = (y - \text{K})$. Our equation then becomes:

$$x\text{Y} = \text{C}$$

This is a rectangular hyperbola with the same y-axis but the x-axis shifted K units

upwards, so that when $Y = 0, y = K$. The new origin therefore has the co-ordinates $x = 0, y = K$. If K is not zero, so that the bending point is not equidistant from the origin, we still call our curve a hyperbola, and we can meet the equation of a hyperbola in a more complicated disguise. Chart 81 shows one with 3 constants:

$$y = \frac{3x + 10}{x - 2}$$

On the face of it this does not look like our standard form, but we find that it is essentially so, if we graph a series of tabulated values such as:

$x = -14, \quad -6, \quad -2, \quad 0, \quad +1, \quad +2, \quad +3, \quad +4, \quad +6, \quad +10, \quad +18 \ldots$
$y = \quad +2, \quad +1, \quad -1, \quad -5, \quad -13, \quad \infty, \quad +19, \quad +11, \quad +7, \quad +5, \quad +4 \ldots$

Let us now shift our axes, starting with the y-axis. Put $(x - 2) = X$. The equation now becomes:

$$y = \frac{3(X + 2) + 10}{X} = \frac{3X + 16}{X} = 3 + \frac{16}{X}$$

$$\therefore \quad (y - 3) = \frac{16}{X}$$

We have shifted our y-axis, 2 units to the right, so that $X = 0$ when $x = 2$. We now shift the x-axis by putting $Y = (y - 3)$. Our equation referred to the new grid becomes:

$$Y = \frac{16}{X}$$

In other words, the graph of the original equation is identical with a rectangular hyperbola whose knee joint is at $X = 4 = Y$ or $X = -4 = Y$, when we reckon X and Y from a new set of axes parallel to the old ones, but crossing at $x = 2$, $y = 3$. With reference to the new origin, our graph is that of a rectangular hyperbola, and its limbs are *asymptotic* to the new axes. We can therefore see the necessary transformation by looking for the y-co-ordinate of the line to which the horizontal limbs are asymptotic and the x-co-ordinate of the line to which the vertical limbs are asymptotic.

Ex. 12.22

Plot the following. Find the origin with reference to which the hyperbola is rectangular by inspection of the asymptotes and by algebraic substitution as above:

1. $y = \dfrac{4x - 5}{x + 3}$ 2. $y = \dfrac{2x + 3}{x + 1}$

3. $y = \dfrac{x + 2}{3x + 4}$ 4. $y = \dfrac{10x + 2}{5x - 1}$

* * * * *

In the denominator of the example which preceded the last exercise the coefficient of x was unity. In the last 2 exercises, we introduce another constant. The introduction of the new constant simply changes the scale of measurement along one axis. A peculiar property of the hyperbola is that change of scale of x measurements alone has just the same result as a corresponding change of scale of y-measurements alone. Let us go back to the hyperbola

$$xy = 16$$

Suppose we halve the value of our x measurements, so that 10 s.p.u. which stand for 10 on the original (x-scale) are equivalent to 5 on the new one (X-scale). Thus the point $x = 4, y = 4$ on our old graph we now label $X = 2, y = 4$. In algebraic language we now have $x = 2X$ and our equation is $Xy = 8$. If we had halved the value of our y-measurements, we should have to put $y = 2Y$. Hence our equation would become: $xY = 8$. The effect is the same either way. It brings the knee-joint of the hyperbola nearer the origin. On the original grid its co-ordinates are $x = \pm 4 = y$. On the new grid its co-ordinates are $X = \sqrt{8} = y$ or $x = \sqrt{8} = Y$. Change of scale does not affect the *symmetry* of the graph, as you might be tempted to suppose.

Ex. 12.23

Plot each of the graphs in Ex. 12.12 on the same grid with 4 scales: (*a*) 1 s.p.u. for $x = 1 = y$; (*b*) 1 s.p.u. for $x = 1$, 3 s.p.u. for $y = 1$; (*c*) 1 s.p.u. for $y = 1$, 3 s.p.u. for $x = 1$; (*c*) 3 s.p.u. for $x = 1 = y$.

* * * * *

§ 3. GEOMETRICAL REPRESENTATION OF INVERSE PROPORTION

Chart 82 shows another visualization of inverse proportion. It is neither a graph nor a histogram, though it looks somewhat like the latter. It is a composite picture of a large number of rectangles, the areas of which are all the same. An

area is the product of two measurements, its vertical *height* and its *breadth*. We can represent this by the formula:

$$A = h \, . \, b$$

If we are merely told what is the value of A, we cannot draw a rectangle. All we know about it is that it is a member of an indefinitely large family of oblong shapes connected by the fact that two particular measurements have a fixed product, and are therefore inversely proportional. If we superimpose them, with two sides always in alignment, we get a figure of which the contour is a rectangular hyperbola. All the free corners lie on the latter. The free corner of the square, when $h = b = \sqrt{A}$, is its knee-joint.

Ex. 12.31

Draw diagrams like those of Chart 82 for rectangles of area

1. 16 sq. cm. 2. 36 sq. cm. 3. 25 sq. cm.

* * * * *

In accordance with the convention of Chart 73, Chart 82 shows two ways of visualizing a *positive* area; but if we are entitled to speak of a height below ground level as negative or a breadth left of a fixed point as negative, we are entitled to adopt a convention which makes areas of rectangles negative if their adjacent sides have opposite signs (either $-x$ and $+y$ or $+x$ and $-y$). So far we have confined ourselves to hyperbolas with two separate portions respectively lying above one fixed line to the right of a second at right angles to it and below the first to the left of the other. This is because C is positive, as we usually imply when we say that y is inversely proportional to x. The effect of introducing a negative sign in front of C is to transpose the two wings, so that they lie in the alternative quadrants. For instance, $xy = -16$, x is always negative when y is positive and *vice versa*. The two wings thus lie in the left upper and right lower quadrants.

Ex. 12.32

Plot the following.

1. $xy = -36$ 2. $x = \dfrac{-50}{y + 10}$

3. $y = \dfrac{4 - 10x}{x - 2}$

§ 4. INVERSE PROPORTION AND HARMONIC SERIES

Elementary science provides several examples of inverse proportion of the sort which involves *continuous* change of both variables. When we double the pressure on a piston we halve the volume of air enclosed, and we can change either the pressure or the volume of a gas as gradually as we like. There is no discontinuity. So the histogram of Chart 79 does not describe such a situation. Though there are many familiar examples of *direct* proportion involving growth by jumps (e.g. *price* problems), no such examples of inverse proportion come within the range of our daily experience. We meet them in some branches of scientific research; for instance, the weeding out of certain types of annual plants by selective breeding. The proportions of such types may be inversely proportional to the number of generations, necessarily a whole number.

The name for series of terms which are inversely proportional to whole numbers is *Harmonic Series*. The reason for the name depends on a discovery made in the sixth century B.C. by the Pythagoreans (Chapter 4, Book I). When we pluck a filament of particular material and thickness and tension, fixed at both ends, it emits a note, the pitch of which depends on its length. Other things (material, etc.) being equal, a longer string gives a note which is of lower pitch. Halving the length raises the pitch an octave, or as we now say doubles its frequency. The Greeks, who made the first recorded studies on the proper design of musical instruments, discovered that the notes of strings *harmonize* when the lengths form the reciprocals of a whole number series. For example, if the longest has 1 unit of length, the note given out by it will harmonize with that given out by similar strings $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ units long. Now look at the following:

$$n = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \ldots$$

$$H_n = \frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \ldots$$

The series of the second line is a harmonic series or harmonic progression of the simplest sort:

$$H_n = \frac{1}{n} \quad \textit{or} \quad n \,.\, H_n = 1$$

The numerator of H_n remains the same, while the *denominator goes up by equal steps*. A harmonic series is therefore a series of which the reciprocals arranged in the same order make up an arithmetic series. For instance, here are 3 other examples:

$$\frac{10}{1} \quad \frac{10}{3} \quad \frac{10}{5} \quad \frac{10}{7} \quad \frac{10}{9} \ldots$$

$$\frac{4}{5} \quad \frac{2}{5} \quad \frac{4}{15} \quad \frac{1}{5} \quad \frac{4}{25} \ldots$$

$$2 \quad 1\tfrac{5}{7} \quad 1\tfrac{1}{2} \quad 1\tfrac{1}{3} \quad 1\tfrac{1}{5} \ldots$$

It is not obvious that the last two are harmonic series, till we arrange them so that the numerators are the same:

$$\frac{4}{5} \quad \frac{4}{10} \quad \frac{4}{15} \quad \frac{4}{20} \quad \frac{4}{25} \ldots$$

$$\frac{12}{6} \quad \frac{12}{7} \quad \frac{12}{8} \quad \frac{12}{9} \quad \frac{12}{10} \ldots$$

To test whether a series is an H.P. you can: (*a*) invert each term to get its reciprocal, and test the converted series to see if it is an A.P.; (*b*) get each term into a form with the same numerator and see whether the denominators go up by equal steps.

Example 1

$$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$
$$H_n = \tfrac{1}{4} \quad \tfrac{2}{7} \quad \tfrac{3}{10} \quad \tfrac{4}{13} \quad \tfrac{5}{16} \quad \tfrac{6}{19}$$

The L.C.M. of the numerators is 60. We can therefore apply the second method thus:

$$H_n = \frac{60}{240} \quad \frac{60}{210} \quad \frac{60}{200} \quad \frac{60}{195} \quad \frac{60}{192} \quad \frac{60}{190}$$

Clearly the denominators are *not* an A.P. So this series is *not* an H.P.

Example 2

$$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$
$$H_n = \tfrac{1}{3} \quad \tfrac{1}{8} \quad \tfrac{4}{52} \quad \tfrac{3}{54} \quad \tfrac{2}{46} \quad \tfrac{2}{56}$$

We get by inversion the series:

$$3 \quad 8 \quad 13 \quad 18 \quad 23 \quad 28$$

This is an A.P. of common difference 5, the general formula being:

$$A_n = A_0 + 5n = 3 + 5n$$

$$\therefore \quad H_n = \frac{1}{3 + 5n}$$

Ex. 12.41

Use the method of the last 2 examples to test which of the following are harmonic series:

1. $3 \quad 11 \quad 19 \quad 27 \quad 35 \ldots$
2. $96 \quad 48 \quad 32 \quad 24 \quad 19\frac{1}{5} \ldots$
3. $\frac{4}{5} \quad \frac{24}{35} \quad \frac{3}{5} \quad \frac{8}{15} \quad \frac{12}{25} \ldots$
4. $\frac{3}{2} \quad \frac{5}{3} \quad 1 \quad \frac{3}{4} \quad \frac{1}{2} \ldots$
5. $1\frac{1}{3} \quad \frac{2}{3} \quad \frac{4}{9} \quad \frac{1}{3} \quad \frac{4}{15} \ldots$

* * * * *

There is another test which depends on the same definition. If a, b, c are successive terms of an H.P., we have seen that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are successive terms of an A.P.

$$\therefore \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\therefore \quad \frac{a - b}{ab} = \frac{b - c}{bc}$$

$$\therefore \quad c(a - b) = a(b - c)$$

$$\therefore \quad ac - bc = ab - ac$$

$$\therefore \quad 2ac = ab + bc$$

$$\therefore \quad b = \frac{2ac}{a + c}$$

The middle term b of three consecutive terms of an H.P. is called the *Harmonic* Mean. The formula for the harmonic mean is a way of testing a series to find whether it is harmonic.

Example:

$$5, \frac{5}{8}, \frac{1}{3}, \frac{5}{22}, \frac{10}{58} \ldots$$

Pick on any two terms separated by only one other, e.g. $\frac{1}{3}$ and $\frac{10}{58}$. If the series is an H.P. the intervening term ($\frac{5}{22}$) is the harmonic mean. From the formula the latter is:

$$\frac{2(\frac{1}{3} \cdot \frac{10}{58})}{\frac{1}{3} + \frac{10}{58}} = \frac{20}{88} = \frac{5}{22}$$

Similarly we may pick on 5 and $\frac{1}{3}$. We then get:

$$\frac{2(5 \cdot \frac{1}{3})}{5 + \frac{1}{3}} = \frac{5}{8}$$

This is again in agreement with the data.

Ex. 12.42

Find the Harmonic Mean of the following numbers:

1. 3 *and* 5. 2. 7 *and* 0.5. 3. -1 *and* $+4$
4. -3 *and* $+1$. 5. 5 *and* 14. 6. 7 *and* 12.

* * * * *

General Formulae for Harmonic Series. We have seen that H_0, H_1, H_2, etc., make up a harmonic series, if the following make up an arithmetic series:

$$A_0 = \frac{1}{H_0};\ A_1 = \frac{1}{H_1};\ A_2 = \frac{1}{H_2}\ \ldots\ A_n = \frac{1}{H_n}$$

Conversely, we may write:

$$H_0 = \frac{1}{A_0};\ H_1 = \frac{1}{A_1}\ \ldots\ H_n = \frac{1}{A_n}$$

We shall call A_n the *parent* series. Now if d is the common difference of the parent series:

$$\begin{aligned} A_n &= A_0 + nd \\ &= \frac{1}{H_0} + nd \\ &= \frac{1 + n \cdot d \cdot H_0}{H_0} \end{aligned}$$

$$\therefore\ H_n = \frac{1}{A_n} = \frac{H_0}{1 + ndH_0}$$

For any H.P. d and H_0 are fixed. So the product is also fixed. We may call this product K, i.e.

$$H_n = \frac{H_0}{1 + K \,.\, n}$$

Example. Find a formula for the series

$$5 \quad \tfrac{5}{8} \quad \tfrac{1}{3} \quad \tfrac{5}{22} \quad \tfrac{10}{58}$$

We have already found that this is an H.P. On inversion, we get the parent series:

$$\tfrac{1}{5},\ \tfrac{8}{5},\ \tfrac{15}{5},\ \tfrac{22}{5},\ \tfrac{29}{5}$$

Here $A_0 = \frac{1}{5}$ and $d = \frac{7}{5}$.

$$\therefore\quad A_n = \frac{1}{5} + \frac{7n}{5} = \frac{1 + 7n}{5}$$

$$\therefore\quad H_n = \frac{5}{1 + 7n}$$

Ex. 12.43

Find a formula for the following harmonic series:

1. $1\frac{1}{2},\ \frac{1}{3},\ \frac{3}{16},\ \frac{3}{23},\ \frac{1}{10}\ \ldots$
2. $\frac{2}{3},\ \frac{2}{7},\ \frac{2}{11},\ \frac{2}{15},\ \frac{2}{19}\ \ldots$
3. $1\frac{1}{2},\ \frac{3}{5},\ \frac{3}{8},\ \frac{12}{44},\ \frac{6}{28}\ \ldots$
4. $2,\ \frac{4}{7},\ \frac{1}{3},\ \frac{8}{34},\ \frac{2}{11}\ \ldots$
5. $\frac{7}{11},\ \frac{1}{2},\ \frac{14}{34},\ \frac{7}{20},\ \frac{17}{46}\ \ldots$

* * * * *

**Difference Formulae*. We have previously learnt to use the difference symbol Δ for the amount by which F_n increases when we step up n by one, i.e. $\Delta F_n = F_{n+1} - F_n$. Arithmetic progressions increase by equal steps of d (the common difference), i.e.

$$\Delta A_n = d$$

We can obtain a difference equation for harmonic series from the definition of H_n:

$$H_n = \frac{H_0}{1 + K \,.\, n} \qquad \therefore\quad H_{n+1} = \frac{H_0}{1 + K(n+1)}$$

$$\therefore\quad \Delta H_n = \frac{H_0}{1 + K(n+1)} - \frac{H_0}{1 + Kn}$$

$$= \frac{-K \,.\, H_0}{(1 + K + Kn)(1 + Kn)}$$

Since K and H_0 are fixed numbers for a particular H.P. we may replace $1 + K$ and KH_0 by two constants A, B, i.e.

$$\Delta H_n = \frac{-A}{(B + Kn)(1 + Kn)}$$

Ex. 12.44

Test the Δ formula by recourse to the harmonic series of Ex. 12.43.

*§ 5. ROTATING THE AXES

We have now seen how the formula for one and the same curve, the *hyperbola*, changes when we shift the y-axis of the grid to the right or to the left and the x-axis upwards or downwards without changing the direction. The various formulae we have derived do not exhaust all the guises in which we may meet a hyperbola. The following exercise shows you another set.

Ex. 12.51

Plot the following functions, draw the asymptotes and investigate their relation to the grid.

1. $y = \sqrt{x^2 - 2}$
2. $y = \sqrt{x^2 - 8}$
3. $y = \sqrt{x^2 - 18}$
4. $y = \sqrt{x^2 - 32}$
5. $y = \sqrt{x^2 - 50}$

CHAPTER 13

Puzzle Corner

§ 1. PROBLEMS IN GENERAL AND SHARING PROBLEMS IN PARTICULAR

Man seems to be the only living creature who deliberately sets himself problems, getting entertainment from solving them. There are few newspapers or magazines which do not contain some kind of quiz, brain-teaser, conundrum, cross-word puzzle or other form of intellectual entertainment, and their variety is legion. The only practical difference between puzzles and other problems is that puzzles usually have an element of quaintness, humour or fantasy. They are tasks without tears.

TRANSLATION

Algebra is a special kind of language; but most puzzles are expressed in everyday language. If we are going to solve them by Algebra we must therefore be able to translate English sentences into Algebra sentences. This is the whole secret of algebraic puzzle-solving. The words used in everyday language follow certain rules which we call *grammar*. Algebra also has its grammar. We have already learnt a good deal of it. We know the words and grammar of English and we know the signs and rules of Algebra. All we now need is skill in translating from one to the other. Algebra is a *special* language because one cannot use it to say anything whatever. We cannot say: *it is a fine day* in Algebra; but we *can* say: *the temperature on August 1st is eighty degrees Fahrenheit*, e.g.: $T_{1.8} = 80°$. We cannot say: *Harry is more generous than Mary*; but we can say: *Harry gives away more than Mary*, if we know the signs $>$ (*greater than*) or $<$ (*less than*). If S_H mean the sum of Harry's gifts to charity and S_M mean Mary's, we can write $S_H > S_M$.

Generosity is a *quality*. A sum of money is a *quantity*. We cannot add or divide qualities; and Algebra is concerned only with quantities. We can use Algebra language to answer such questions as: *how many?*, *how much?*, *how big?*, *when?* (i.e. how long after a given time?), *where?* (i.e. how far from a given point?). It cannot answer such questions as: *why?*, *what sort of?*, *how?* In our ordinary language we distinguish broadly between things and actions. Thus we have *nouns* and *verbs*.

The other parts of speech simply help to support these two essentials. Thus adjectives tell us more about the noun. Adverbs tell us more about the verbs. Conjunctions join words and sentences. Now consider the algebraic counterparts of these. Take a simple descriptive statement first. In quality language we can be quite vague, if we wish. In the quantity language called Algebra we must be more precise. If we agree that a *long* road (R) is one of *more than ten miles*, we can write out the following statement in Algebra:

Quality Language	*Quantity* Language
THE WAY WAS LONG	$R > 10$

If we define a *cold* wind as a wind whose temperature is below freezing-point, an *infirm* person as a person of medical category C.3, and an old one as over 65, we can translate:

The wind was cold	$T_w < 0°$ C.
The minstrel was infirm	$H_M = C.3$
and old	$A_M > 65$

Ex. 13.11. SIMPLE TRANSLATIONS

Express the following in suitable (i.e. *subscript*) algebraic symbolism. Where necessary make the conditions more than precise.

1. Father's age is 50.
2. Mother is younger than Father.
3. The box is five feet long.
4. The length is greater than the breadth.
5. John has sixpence more than Edith.
6. The bridge could not take a heavy load.
7. The water is boiling hot.
8. The giraffe was eight times as tall as the monkey.
9. The journey took 15 hours.
10. The rate at which the bacteria reproduce themselves is 8 times a day.

* * * * *

Before translating we may thus have to restate the original sentence, in order to make its meaning clearer. The problem must be broken down into simple statements which can be directly expressed in Algebra. One peculiarity of the

language of Algebra is that you can choose what symbols you like. They can be any letters of our own alphabet, or in any foreign alphabet. We can make up new symbols, though it does save effort to choose symbols (like A_f for father's age) sufficiently suggestive to remind you about what you are really discussing. What you *must* do is to obey the rules of algebraic grammar. The only signs which are fixed are such signs as $+$, $\times$, $=$. These correspond to *verbs* in quality language. For example, take the simple rule for finding the area of a rectangle:

Multiply length by breadth to get area.

We express this as follows:

$l \times b = A$, i.e.

The length	l
must be multiplied by	$\times$
the breadth	b
to get	$=$
the area	A

In the above equation the signs l, b, A, all correspond to *nouns*. The signs $\times$, $=$, correspond to *verbs*. Sometimes we have to reckon with adjectives and adverbs, or phrases which serve as adjectives and adverbs, e.g. *the speed of the third train* might be translated: s_3; the *minstrel's age:* A_m. We have used suffixes all the way through this book to distinguish one quantity from another one like it. These suffixes correspond to adjectives. Examples of adverbial signs are the *super* and *sub*script letters attached to $\sum\limits_1^n$, meaning: ADD UP *for all values from* 1 *to n*. We have translated in different places the sign for *equality* ($=$) in two senses. It means *is* in some sentences and *to get* in others. The difference is more apparent than real. We can equally well say: *multiply the length by the breadth to get the area* or *the length multiplied by the breadth is the area.*

We can add algebraic sentences together:

Mr. Jones has two sons:	$s_J = 2$
Mr. Brown has three sons:	$s_B = 3$
Mr. Jones and Mr. Brown together have five sons:	$s_J + s_B = (2 + 3)$

We have added s_B to the left-hand side of the first equation and 3 to the right, and this is justified because s_B and 3 are equal. In *quality* language we cannot always do this. Look at these two sentences: (*a*) this mouse is a mammal, (*b*) this

mouse is big. These do not add up to the statement: (*c*) this mouse is a big mammal. This assertion is nonsense because the word *big* by itself has no precise meaning. If we substitute *six inches long*, we can correctly say: this mouse is a six-inch mammal. A great deal of puzzle solving depends on combining statements together, and you must always take care that the meaning of a statement is perfectly clear before you try to translate it.

The Three Steps

We shall begin with a simple problem about sharing money, and show the translation and solution step by step.

Example:

> *Divide* 2/6 *between Osbert and Gilbert so that Osbert has half as much again as Gilbert.*

1st step: Choice of symbols and units:

Let Osbert's share $= x$, Gilbert's $= y$. We cannot have *mixed* units in a problem. This is a rule of Algebra grammar. So we reduce to pence. 2/6 = 30d.

2nd step: Translation of statements.

This often means reframing the original statement. As it stands we cannot *divide* 2/6. We do not know what to divide it by. On reflection we see that Osbert and Gilbert have between them 30d. Thus the sum of their shares is 30d. This is a statement we can translate directly into Algebra. Thus:

$$x + y = 30 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Now Osbert's share is half as much again as Gilbert's, i.e. $1\frac{1}{2}$ times as large. Thus:

$$x = 1\tfrac{1}{2}y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

3rd step: Solution of equations.

We have to find one unknown at a time. This means getting rid of the other. We have already learnt how to *eliminate by substitution*. In equation (1) we replace x by $1\frac{1}{2}y$.
Thus:

$$1\tfrac{1}{2}y + y = 30 \qquad \therefore \; 2\tfrac{1}{2}y = 30$$
$$\therefore \; 5y = 60 \qquad \therefore \; y = 12$$

Now replace y by 12 in (1).

$$x + 12 = 30 \quad \therefore \quad x = 18$$

Answer: Osbert has 1/6 (18 pence); Gilbert has 1/– (12 pence)

Check: $1/6 + 1/- = 2/6$; $1/6 = 1\frac{1}{2}$ (1/–)

Of course this was a very simple problem which you could do more quickly in your head; but the same three steps—*symbolism*, *translation*, *solution*—must always be followed.

Ex. 13.12. Simple Sharing Puzzles

1. Share a guinea between two doctors so that Dr. Bone gets six times as much as Dr. Blood.
2. Share a ton of coal between Mrs. Black and Mrs. White so that Mrs. Black gets two-thirds as much as Mrs. White.
3. Share £4 8s. between two solicitors so that Mr. Twist gets one-fifth as much again as Mr. Twiddle.
4. Two undertakers, Mr. Moan and Mr. Groan, share the profit on a funeral so that Mr. Moan gets £8 more than Mr. Groan. This makes Mr. Groan's share equal to three-quarters of Mr. Moan's. How much did each make?
5. The appetites of Charlie the Chimpanzee and Bill the Baboon are always in the ratio of 3 to 7. How would they divide 70 nuts between them?
6. Two sailors, Sam Salt and Bertie Breeze, steal some bottles of rum. Sam finds he has three times as many bottles as Bertie so he gives him 12 bottles and they have then an equal number. How many bottles do they steal between them?
7. Two hikers, Short and Long, share a packet of sandwiches. Short eats two more than Long. If Short had eaten one more and Long one less, Short would have had twice as many as Long. How many sandwiches were in the packet?
8. Divide 80 into two parts such that the one part is the square of half the other part.
9. The square of a positive number, added to twice its cube makes 21 times the number. What is it?
10. Divide 90 into two parts such that half the one part is one-thirteenth of the other.

§ 2. PROBLEMS WITH SEVERAL CONDITIONS

We can now apply these ideas to a longer puzzle. Example: *Divide 45/- between Amy, Betty, Caroline and Daphne so that if we add 2/- to Amy's share, subtract 2/- from Betty's share, multiply Caroline's share by 2, or divide Daphne's share by 2, the result is the same in every case.*

1st step: Choose the symbols. Let the four shares be a, b, c, d respectively.

2nd step: Translate all the information. We know that the four shares add up to 45. Thus:

$$a + b + c + d = 45$$

We also know that:

Amy's share plus 2/-	$a + 2$
equals	$=$
Betty's share minus 2/-	$b - 2$
equals	$=$
Caroline's share multiplied by 2	$c \times 2$
equals	$=$
Daphne's share divided by 2	$d \div 2$

Each *equals* sign ($=$) gives us a separate equation. It is important to remember that the number of equations must be as great as the number of unknown quantities.

3rd step: Solve the equations. This means finding the value of each letter separately. Since each equation has more than one letter, we must juggle with each equation till we get rid of all except one letter. It is best to start with the shorter equations.

$$a + 2 = b - 2 \qquad \therefore \quad a + 4 = b$$

$$\text{Next } b - 2 = c \times 2 \qquad \therefore \quad 2c = a + 2$$

$$\therefore \quad c = \tfrac{1}{2}(a + 2)$$

$$\text{Next } c \times 2 = d \div 2 \qquad \therefore \quad \frac{d}{2} = a + 2$$

$$\therefore \quad d = 2(a + 2)$$

Thus we have expressed b, c, d in terms of a; but

$$a + b + c + d = 45$$

$$\therefore \quad a + (a + 4) + \tfrac{1}{2}(a + 2) + 2(a + 2) = 45$$

Multiply all through by 2 to get rid of fractions, so that:

$$2a + 2a + 8 + a + 2 + 4a + 8 = 90$$

Collect like terms:

$$(2a + 2a + a + 4a) + (8 + 2 + 8) = 90$$
$$9a + 18 = 90$$
$$\therefore\ 9a = 72 \qquad \therefore\ a = 8$$

Now $a + 4 = b \qquad \therefore\ b = 12$

$$c = \tfrac{1}{2}(a + 2) = \tfrac{1}{2}(8 + 2) \qquad \therefore\ c = 5$$
$$d = 2(a + 2) = 2(8 + 2) \qquad \therefore\ d = 20$$

Answer: Amy 8/– Betty 12/– Caroline 5/– Daphne 20/–

Check: $8 + 12 + 5 + 20 = 45$; $8 + 2 = 12 - 2 = 5 \times 2 = 20 \div 2$

. . . . .

Next is an example of a puzzle involving a quadratic equation. *Ivan, Igor and Ivanoff share a certain number of roubles. Ivanoff has as much more than Igor as Igor has than Ivan. The product of Ivan's share with Igor's equals the total amount. Ivanoff has one more rouble than Ivan and Igor together. How many roubles has each?*

1st step: Let the three shares be x, y, z.

2nd step:

1. How much more Ivanoff has than Igor — $(z - y)$
   is the same as — $=$
   how much more Igor has than Ivan — $(y - x)$
   i.e.

$$z - y = y - x$$
or
$$x - 2y + z = 0 \quad \ldots\ldots \quad (1)$$

2. The product of Ivan's share with Igor's — xy
   equals — $=$
   the total amount — $x + y + z$
   i.e.

$$xy - x - y - z = 0 \quad \ldots\ldots \quad (2)$$

3. Ivanoff — z
   has — $=$
   one more rouble than Ivan and Igor together — $1 + x + y$
   i.e.

$$z = 1 + x + y \quad \ldots\ldots \quad (3)$$

3rd step: By means of (3) we can eliminate z.

Replace z in (1) by $(1 + x + y)$

(1) becomes

$$x - 2y + 1 + x + y = 0$$
$$\therefore \quad 2x - y + 1 = 0$$
$$\therefore \quad y = 2x + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By means of (4) we can eliminate y, i.e. replace y in (2) by $2x + 1$. Thus (2) becomes

$$x(2x + 1) - x - (2x + 1) - z = 0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We can also replace z by means of (3). We then have:

$$z = 1 + x + y = 1 + x + 2x + 1 = 2 + 3x$$

So (5) now becomes:

$$x(2x + 1) - x - (2x + 1) - (2 + 3x) = 0$$
$$\therefore \quad 2x^2 + x - x - 2x - 1 - 2 - 3x = 0$$
$$\therefore \quad 2x^2 - 5x - 3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We can solve (6) either by factors or by formula or by graph. The quickest solution is by factors. By factorizing we have:

$$(x - 3)(2x + 1) = 0$$
$$\therefore \quad x = +3 \text{ or } -\tfrac{1}{2}$$

In this puzzle the negative root is meaningless for our purpose. So we adopt $+3$ as the value for x, i.e. Ivan's share is 3 roubles. We easily find y and z from equations (4) and (3):

$$y = 2x + 1 = 2 . 3 + 1 = 7$$
$$z = 1 + x + y = 1 + 3 + 7 = 11$$

Answer: Ivan 3 roubles; Igor 7 roubles; Ivanoff 11 roubles.

Check:

$$\begin{aligned}
\text{(Ivanoff's share)} - \text{(Igor's share)} &= z - y \\
&= 11 - 7 = 4 \\
\text{(Igor's share)} - \text{(Ivan's share)} &= y - x \\
&= 7 - 3 = 4
\end{aligned}$$

$$\text{(Ivanoff's share)} = 11 = 1 + 10 = 1 + (3 + 7)$$

which is one more than Ivan's and Igor's shares together.

REMINDER

A number of special arithmetical terms continually recur. You have learnt them all; but it is useful to have them all together in one list.

The *sum* of x and y is $(x + y)$.

The *difference* of x and y is $(x - y)$ or $(y - x)$.

The *product* of x and y is $x \times y = xy = x \,.\, y = x(y)$

The *quotient* of x by y is $x \div y$, i.e. $\frac{x}{y}$.

The *digits* in a number are the unit values of the figures. Thus 3582 means $3000 + 500 + 80 + 2$ but the digits are simply 3, 5, 8, 2. *Thus the sum of the digits* is $3 + 5 + 8 + 2 = 18$. *The product of the digits* is $3 \times 5 \times 8 \times 2 = 240$.

Half as much again means $1\frac{1}{2}$ times.

As much again means *twice*.

A *multiple* of x means nx in which n is a whole number.

An *integer* means a whole number, e.g. 7, 19, 24.

A *factor* means an integer which divides into another integer without a remainder.

A *prime* (or *prime factor* or prime number) is one whose only factors are itself and 1, e.g. 2, 17, 23.

Unity means 1 (one).

Even numbers are those which are divisible by 2, e.g. 2, 12, 30.

Odd numbers are not divisible by 2, e.g. 5, 19.

Consecutive numbers are two or more numbers each differing from its neighbours by unity, e.g. 6, 7, 8, 9.

The square of a number means the product of a number and itself, e.g.

$$4^2 = 4 \times 4 = 16$$
$$x^2 = x \times x$$

The *cube of a number* means the repeated product of a number with itself and itself again, e.g.

$$5^3 = 5 \times 5 \times 5 = 25$$
$$x^3 = x \times x \times x$$

An *operation* means any algebraic process which we perform on a number, e.g. multiplication.

The *inverse* of an operation means working back from the result of an operation to the starting-number, e.g. the inverse of the operation $5 \times 3 = 15$ is $15 \div 3 = 5$. This division is the inverse of multiplication. Subtraction is the inverse of addition. The *inverse of the inverse* gives us the original operation, e.g. $(15 \div 3) \times 3 = 15$.

The *square root* is the inverse of the square and the *cube root* is the inverse of the cube, e.g. $\sqrt{9} = 3$ and $\sqrt[3]{64} = 4$.

Ex. 13.21. Harder Mixed Problems

1. One-third of the population of a town are boys and girls. Of these 53 out of every 100 are girls. If the number of boys is 2,068, what is the whole population?
2. In Rugby football a *try* counts 3 points and a *goal* counts 5 points. If one team scores tries only and the opposing scores goals only, what is the least score for a draw?
3. Wales beats Scotland at rugger by 4 points. Wales scores half as many goals again as Scotland. Scotland scores half as many tries again as Wales. How many goals and tries did each score and what were the totals?
4. Egbert and Osbert are playing a card game. The stakes are beans. They start with equal numbers. Egbert wins 18 beans from Osbert. Then Osbert wins enough to double what he has left. Then Egbert wins as many as he has left. Egbert now has as many dozens as Osbert has beans. How many beans has each at the end?
5. What year was this: the four digits add up to 21: the second digit is the sum of the first and fourth: half the sum of the first, second and fourth equals the third which is also equal to the sum of the fourth and first?
6. In a family of sons and daughters each son has twice as many sisters as he has brothers, but each daughter has just as many brothers as sisters. How many of each are there?
7. What odd multiple of 5, less than 100, equals 5 times the sum of its digits?
8. I have a square cabbage patch, the cabbages being equally spaced both ways. I remove the outer row all round the square and count them. There are 332. How many cabbages are there?
9. A shepherd puts his sheep in three pens. In one there were six more than a quarter of the flock. In the next were three more than half the remainder. In the third pen were six more than in the first. How many sheep had he altogether?

10. Of six consecutive whole numbers the product of the last three exceeds the product of the first three by 816. Find the numbers.

Ex. 13.22. NUMBER PUZZLES

1. A certain number when diminished by 1 and multiplied by 12 gives one less than the square of the number. Find the number.
2. The sum of the half and the quarter of a number give the cube of the quarter of the third of the number. Find the number.
3. The quotient of the square of the sum of the digits of a 2-figure number by half the square root of the number gives 25. Find the number.
4. The sum of the digits of a 3-figure number is the product of the last two digits. The first, third and second digits, when rearranged in that order are consecutive. Find the number.
5. The famous mathematician, Augustus de Morgan, was x years old in the year x^2. He died in 1871. When was he born?
6. The sum of the digits of a 2-figure number is 12. The square of the second digit equals the cube of the first. Find the number.
7. The product of three consecutive integers is one-twentieth of the product of the next three consecutive integers. Find the six integers.
8. Find a number the square of whose half equals the cube of its third.
9. Find the sum of all numbers less than 100 which are not divisible by 3.
10. What is the least whole number by which $1{,}925 \times 3{,}465 \times 4{,}032$ must be multiplied to give a perfect square?

§ 3. PROPORTION

There is a rich crop of trench-digging, hay-eating, tank-filling problems, all concerned with *proportion*.

The basic rules of proportion can be combined in an equation.

If Q_n is *directly* proportional to a_n, b_n, c_n, d_n, etc., and *inversely* to w_n, x_n, y_n, z_n:

$$Q_n = K\frac{a_n \cdot b_n \cdot c_n \cdot d_n \ldots}{w_n \cdot x_n \cdot y_n \cdot z_n \ldots}$$

Similarly:

$$Q_m = K\frac{a_m \cdot b_m \cdot c_m \cdot d_m \ldots}{w_m \cdot x_m \cdot y_m \cdot z_m \ldots}$$

If we combine the above, we get the most general rule:

$$\frac{Q_n}{Q_m}=\frac{a_n \cdot b_n \cdot c_n \cdot d_n \ldots}{a_m \cdot b_m \cdot c_m \cdot d_m \ldots}\times\frac{w_m \cdot x_m \cdot y_m \cdot z_m \ldots}{w_n \cdot x_n \cdot y_n \cdot z_n \ldots}$$

To apply the rule we have merely to sort out the various quantities involved into two groups, having direct and inverse proportion respectively, and construct our equation accordingly. This is a hard example:

> 100 *men, in* 6 *days of* 10 *hours each, dig a trench* 200 *yards long,* 3 *yards wide,* 2 *yards deep. How many days of* 8 *hours each will it take* 180 *men to dig a trench* 360 *yards long,* 4 *yards wide and* 3 *yards deep, if the ground in the second case is harder than the first in the ratio* 7 : 5, *and* 5 *men of the second gang work as hard as* 6 *men in the first?*

Our symbols with appropriate subscripts will be:

No. of men	$= n$	Length of trench	$= l$
No. of days	$= d$	Width of trench	$= w$
No. of hours	$= h$	Depth of trench	$= t$
Hardness factor	$= r$	Men's strength	$= s$

Given d_{10} we have to find d_8.

Factors making for an *increase* in days required are: l, w, t, r, i.e. d is *directly* proportional to l, w, t, r.

Factors making for a *decrease* in days required are: n, h, s, i.e. d is *inversely* proportional to n, h, s.

The formula is:

$$\frac{d_8}{d_{10}}=\frac{l_8 \cdot w_8 \cdot t_8 \cdot r_8}{l_{10} \cdot w_{10} \cdot t_{10} \cdot r_{10}}\times\frac{n_{10} \cdot h_{10} \cdot s_{10}}{n_8 \cdot h_8 \cdot s_8}$$

Now substitute numerical values:

$$\frac{d_8}{6}=\frac{360 \cdot 4 \cdot 3 \cdot 7}{200 \cdot 3 \cdot 2 \cdot 5}\times\frac{100 \cdot 10 \cdot 6}{180 \cdot 8 \cdot 5}$$

$$\therefore \quad d_8 = 17\tfrac{1}{2} \ (\textit{days}).$$

PERCENTAGES. A percentage is a convenient device for persuading people to deal with decimal fractions without realizing they are doing so. It is important for most people to feel that they are handling manageable numbers, and they feel for some reason that up to, or a little beyond, 100, they are at home. Also

for many purposes an accuracy of about 1 in 100 is good enough. So by using percentages we can deal in whole numbers with quantities which are really fractional. Mathematically there is no point in percentages at all, and decimal fractions are preferable.

1. To express one quantity as a percentage of another. Let the two quantities be a, b.

Then $p = 100\frac{a}{b}$ is the required expression.

e.g. *What percentage is £24 of £32?*

$$p = 100 \cdot \frac{24}{32} = 75\%$$

2. To evaluate a percentage.

e.g. *What is 75 % of £32?* Here we change the formula round:

$$100\frac{a}{b} = p \quad \therefore \quad 100a = pb \quad \therefore \quad a = \frac{pb}{100}$$

Thus
$$a = \frac{£75 \cdot 32}{100} = £24$$

3. To find the quantity which a given quantity is a given percentage of.

e.g. *What quantity is £24 75 % of?* Again changing the formula we have

$$pb = 100a \quad \therefore \quad b = \frac{100a}{p}.$$

Thus
$$b = \frac{£100 \cdot 24}{75} = £32$$

4. To find what is left after losing or gaining a given percentage.

In the above example, if we lose 75% of £32, we lose £24 and are left with

$$32 - 24 = 8, \text{ i.e. } £8$$

We write

$$A_l = b - 75\% \text{ of } b = b - a$$
$$= b - \frac{pb}{100} = b\left(1 - \frac{p}{100}\right)$$

Similarly for a gain:

$$A_g = b\left(1 + \frac{p}{100}\right)$$

5. To find what percentage gain or loss leaves a given amount from another given amount. We have

$$A_g = b\left(1 + \frac{p}{100}\right)$$

$$= b + \frac{bp}{100}$$

$$\therefore \frac{bp}{100} = A_g - b \qquad \therefore bp = 100(A_g - b)$$

$$\therefore p = \frac{100}{b}(A_g - b) \text{ for a gain}$$

$$p = \frac{100}{b}(b - A_l) \text{ for a loss}$$

6. Repeated percentages.

Suppose a quantity loses or gains p %, and the amount left loses or gains p %, and so on repeatedly. What is the final amount? This is the Compound Interest problem and we shall consider it in more detail in a later chapter. Meanwhile here is the formula.

Loss.

$$\text{1st time } {}_1A_l = b\left(1 - \frac{p}{100}\right)$$

$$\text{2nd time } {}_2A_l = {}_1A_l\left(1 - \frac{p}{100}\right) = b\left(1 - \frac{p}{100}\right)^2$$

$$\text{3rd time } {}_3A_l = {}_2A_l\left(1 - \frac{p}{100}\right) = b\left(1 - \frac{p}{100}\right)^3$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$n\text{th time } {}_nA_l = b\left(1 - \frac{p}{100}\right)^n$$

For a gain.

$${}_nA_g = b\left(1 + \frac{p}{100}\right)^n$$

Example.

Air is being pumped out of a flask by an air pump. Each stroke removes 25 % *of the*

air. How much remains in the flask after 6 *strokes, if the original amount of air weighed* 2 *grams?* Our symbols are:

$$b = 2 \quad p = 25 \quad n = 6$$

$$\therefore \quad {}_6A_l = 2\left(1 - \frac{25}{100}\right)^6$$

$$= 2\left(1 - \frac{1}{4}\right)^6$$

$$= 2 \cdot \frac{3^6}{4^6}$$

$$= 2 \cdot \frac{729}{4096} = 0{\cdot}356 \text{ approx.}$$

Thus 0·356 grams of air are left after 6 strokes.

This formula covers a wide range of problems of repeated loss or gain.

Ex. 13.31. Mixed Money Problems

1. William starts a job with a salary of £225 and an annual rise of £7/10/-. Henry starts at £150 but has an annual rise of £20. In how many years will Henry catch up with William?
2. Mr. Brown is a heavy smoker. Over a period of 10 years he spent £300 on tobacco. In the second five years he spent twice as much as during the first five years. What was his average monthly expenditure on tobacco during the first five years?
3. A man has a sum of money, an exact number of pounds. He spends a quarter of it one week, a quarter of what is left the next week, and so on till he has £81 left. How many weeks is it since he started, if he always spends an exact number of pounds?
4. I buy a pair of slippers, two hats and three shirts for £4. The two hats and three shirts cost as much as three pairs of slippers. The three shirts cost as much as the two hats. What is the price of each item?
5. I am buying cutlery. I have 33/-. With this I can buy either 6 forks, or 4 knives and 3 spoons. If I had 3/- more I could buy 8 knives. What is the price of a spoon?
6. A furniture dealer undertakes to supply 200 chairs at 28/- each. He then finds that his stock is not enough, and buys some more. The old stock

cost him 21/– each, but the new lot cost him 32/– each and he loses £18 on the deal. How many chairs did he have to buy?

7. A man leaves half his money to his son Henry, and divides one-third of it between his three daughters Anne, Betty and Caroline in the ratio 3 : 2 : 1. He leaves £500 to a friend and divides the residue between Betty and Caroline in the ratio 4 : 1. If Caroline's total comes to £6,500, how much do each of the others receive?

8. By a man's will his three sons, in the descending order of their ages, get one-third, one-quarter and one-fifth respectively of his money, and the widow gets the rest. The widow, dying not long after, leaves her share to the three sons in equal parts. Altogether the youngest son gets £4,900. What does the eldest son get?

9. A man sells a car for £171 and loses 10% of what he gave for it. How much did he give?

10. £243 is $(75\%)^5$ of how much.

Ex. 13.32. Weights and Measures

1. Two carpets have the same area, 400 sq. ft. One is 5 feet shorter but 4 feet wider than the other. Find the length and breadth of each.

2. The dimensions of a rectangular block are in the proportions 3 : 4 : 5. Its volume is 43,740 cub. in. Find its length, breadth and height.

3. A rectangular courtyard is exactly 24 yd. 1 ft. 4 in. in length and 22 yd. broad, and is to be paved exactly with square stones all the same size. What is the largest size of stone that can be used?

4. Reduce 4 weeks 3 days 17 hours 27 minutes 27 seconds to seconds.

5. Reduce 456,000 cub. ft. to cub. yd.

6. In 385 tons of dynamite there are 2 tons of coal-dust, 260 tons of clay and the rest nitro-glycerine. What is the percentage of nitro-glycerine?

7. A child is weighed, and six months later he is weighed again. His weight is now 69 lb. and he has lost $8\frac{1}{3}\%$. If he had gained $8\frac{1}{3}\%$, what would his weight have been on the second weighing?

8. Three gallons of one spirit contain 7% of water. One gallon of another contains 11% of water. These two are mixed and half a gallon of water added. What is the percentage of pure spirit in the new mixture?

9. A trench 220 yards long, 3 wide and 2 deep, is dug by 240 men working 5 days of 11 hours each. How many days of 9 hours each will it take

24 men to dig a trench 420 yards long, 5 wide and 3 deep, assuming equally hard ground and equally hard work?

10. There are two kinds of gas burners in a house and five of one kind burn as much gas as six of the other in a given time. Five burners of the first kind burning for 5 hours each evening for 10 days cost 4/3. How much will 75 burners of the second kind cost burning for 4 hours every evening for 15 days?

§ 4. PROBLEMS OF TIME, SPACE AND MOTION

One popular class of puzzles depends on the difficulty of dealing with people's ages, unless we refer all our units of time to some fixed point, e.g. the present.

Example 1

Father is 4 times as old as John. Five years ago he was 7 times as old as John was then. How old is each now?

Five years ago everyone was 5 years younger than he is to-day. Thus *ago* is translated by *minus*. Similarly *hence* is translated by *plus*.

Call Father's present age f and John's present age j. Then we can translate the first statement:

$$f = 4j$$

Five years ago Father's age was $(f - 5)$ and John's age was $(j - 5)$. So we can translate the second statement:

$$(f - 5) = 7(j - 5)$$

Thus we have two numbers to find, f and j, and two equations. Throughout both equations f and j mean the ages of Father and John *now*.

$$f = 4j; \qquad (f - 5) = 7(j - 5)$$

We can solve them by getting rid of one number and solving for the other. Put $4j$ in place of f in the second equation.

Then

$$(4j - 5) = 7(j - 5)$$

Solve this equation:

$$4j - 5 = 7j - 35$$

Collect and transpose terms:

$$7j - 4j = 35 - 5$$

$$3j = 30 \qquad \therefore \; j = 10$$

Then since

$$f = 4j \qquad \therefore \; f = 40$$

Father is 40 and John is 10 *now.* Check: Five years ago Father was 35 and John was 5. And $35 = 7 \times 5$.

* * * * *

Example 2

In three years time Harry will be as old as John was three years ago, and if John were three years older he would be three times as old as Henry was when he was three years younger than he is now.

We have here two statements about John's present age (j) and Harry's present age (h): The first is:

Harry's age 3 years hence	$h + 3$	
is the same as	$=$	$\therefore h + 3 = j - 3$
John's age 3 years ago	$j - 3$	

The second statement is:

John's age 3 years hence	$j + 3$	
will be the same as	$=$	$\therefore j + 3 = 3(h - 3)$
three times	$3 \times$	
Harry's age 3 years ago	$h - 3$	

The first equation reduces to:

$$h = j - 6 \text{ or } j = h + 6$$

The second to:

$$3h = j + 12 \text{ or } j = 3h - 12$$

$$\therefore 3h - 12 = h + 6$$

$$\therefore \quad 2h = 18$$

Hence $h = 9$ and $j = 9 + 6 = 15$

Thus Harry is now 9 years of age, and John is 15. Check this yourself.

Chain Statements

Some puzzles take a rot of unravelling. They may contain a long chain of conditions and it is a challenge to your ingenuity to sort them out. In the following the key to the whole thing is that we have to work *backwards.*

Mr. Philemon K. Homer and his wife Baucis are an elderly couple. Philemon is twice as old as Baucis was, when she was twice as old as he was, when he was twice as old as she was, when she was three times as old as he was, when she was born. Baucis is now three times as old as Philemon was when they were married. At that time Baucis was 20. How old is each now? Let

$$p = \text{present age of Philemon}$$
$$b = \text{present age of Baucis}$$

The information given shows that P. is older than B. He was already a certain age when she was born. Hence P. is $(p - b)$ years older than B. This difference holds throughout their lives. Therefore when B. was born, i.e. when her age was 0, P. must have been $(p - b)$ years. Twenty years later, when they were married, B. was 20 and P. must have been $(p - b + 20)$. We are told that B. is now 3 times as old as P. was when they were married. B. is now b years.

$$\therefore \quad b = 3(p - b + 20)$$
$$\therefore \quad b = 3p - 3b + 60$$
$$\therefore \quad 4b = 3p + 60$$
$$\therefore \quad b = \frac{3}{4}p + 15 \qquad (1)$$

Now let us work backwards through the statement, translating it step by step into algebraic terms.

Date	*Statement*	*B.'s age*	*P.'s age*
1.	When she was born, she was:	0	—
	he was:	—	$(p - b)$
2.	When she was 3 times as old as he was, at date (1), she was:	$3(p - b)$	—
3.	When he was twice as old as she was, at date (2), he was:	—	$6(p - b)$
4.	When she was twice as old as he was, at date (3), she was:	$12(p - b)$	—
5.	P. is twice as old as B. was, at date (4), therefore P. is:	—	$24(p - b)$

Thus P.'s present age is $24(p - b)$. But we called his present age p.

$$\therefore p = 24(p - b) \qquad (2)$$

Substitute the value for b obtained in equation (1).

$$p = 24[p - (\tfrac{3}{4}p + 15)] = 24[\tfrac{1}{4}p - 15] = 6p - 24 \cdot 15$$

$$\therefore \quad 5p = 24 \cdot 15 \qquad \therefore \quad p = 24 \cdot 3 \qquad \therefore \quad p = 72$$

$$\therefore \quad 72 = 24(72 - b)$$

$$\therefore \quad 72 = 24 \cdot 72 - 24b$$

$$\therefore \quad 24b = 72(24 - 1) = 72(23)$$

$$\therefore \quad b = 3 \cdot 23 \qquad \therefore \quad b = 69$$

Thus Baucis is 69 and Philemon is 72.

Check this yourself.

Clock Problems.—Let us now express the behaviour of the clock in formulae. First notice that the hour-hand is the really important one. The minute-hand is a refinement. You could tell the time near enough for most purposes from the hour-hand alone. This moves round the dial once in 12 hours, i.e. 360° in 12 hours, i.e. 30° an hour. The space between one hour-mark and the next is divided into 5, marking out minutes for the minute-hand, but the hour-hand moves over these 5 divisions in one hour and therefore takes 12 minutes for each division. The speed of the hour-hand is $\frac{1}{12}$th of the speed of the minute-hand. Each of these minute divisions, being $\frac{1}{5}$th of the hour division, is 6°. Thus:

Speed of hour-hand $= \frac{1}{2}°$ per minute.

i.e. 2 minutes per degree.

Speed of minute-hand $= 6°$ per minute.

i.e. 10 seconds per degree.

Our formulae are as follows:

Let s_H = speed of hour-hand $= \frac{1}{2}°$ per minute

a_H = angle turned through in t minutes

$$\therefore \quad \frac{a_H}{t} = \frac{1}{2} \quad \text{or} \quad a_H = \tfrac{1}{2}t \quad \text{or} \quad t = 2a_H$$

s_M = speed of minute-hand $= 6°$ per minute

$$\therefore \quad \frac{a_M}{t} = 6 \quad \text{or} \quad a_M = 6t \quad \text{or} \quad t = \frac{a_M}{6}$$

Example 1:

When the minute-hand moves through one right angle, how far does the hour-hand move? We can solve this in two ways.

1. Speed of hour-hand $= \frac{1}{12}$ speed of minute-hand.

 $\therefore$ Distance travelled in equal time $= \frac{1}{12}$th.

 $\therefore$ Hour-hand moves $\frac{1}{12}$ of $90°$, i.e. $7\frac{1}{2}°$.

2. In 15 minutes the minute-hand moves through $90°$. In 15 minutes the hour-hand moves $15 \times \frac{1}{2}° = 7\frac{1}{2}$.

Example 2:

How long after 3 o'clock will it be till the minute-hand overtakes the hour-hand?

At 3 o'clock the hands are $90°$ apart. When the hour-hand moves $a_H°$ the minute-hand must move $(90 + a_H)°$

$$\therefore \quad a_M = 90 + a_H$$

$$\therefore \quad 6t = 90 + \tfrac{1}{2}t$$

$$\therefore \quad 5\tfrac{1}{2}t = 90 \qquad \therefore \quad t = \frac{90}{5\frac{1}{2}} = 16\tfrac{4}{11}$$

Thus the two hands coincide at $16\frac{4}{11}$ minutes past 3.

Ex. 13.41. Ages and Clocks

1. A father is three times as old as his son. Five years ago he was five times as old as his son was then. How old was the father when the son was born?
2. When the figures of Mr. Jones's age are reversed we obtain Mrs. Jones's age. The sum of their ages is 11 times the difference and he is the elder. What are their ages?
3. Diophantus passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth and $\frac{1}{7}$ more as a bachelor. Five years after his marriage was born a son who died four years before his father at half his father's age. (Quoted in Vera Sanford's *Short History of Mathematics.*) How long did Diophantus live?
4. In the Rudelsheim family there were two boys and three girls. Ike is twice as old as Ruth. The combined ages of Esther and Ruth are twice the age of Ike. The combined ages of Ike and Solly are twice the combined ages of Ruth and Esther. Hannah is 21. The combined ages of the three girls are twice the combined ages of the two boys. Find the ages of all the children.

5. Jack is twice as old as Jill was when Jack was half as old as Jill will be when Jill is three times as old as Jack was when Jack was three times as old as Jill. The sum of their ages is 44. How old is each?

6. It is 11.15 by the clock. How long is it since the two hands were last exactly in line, and how long will it be till they are in line again?

7. How long after 6 o'clock is it till the hands of the clock coincide?

8. Two clocks are correct at 1 p.m. on Wednesday, December 1, 1943. One loses 8 seconds and the other gains 12 seconds in 24 hours. On what day and at what time by each clock will one be exactly one hour in advance of the other?

9. It is now between 7 and 8 a.m. Between 1 and 2 p.m. this afternoon the hands of my watch will have exactly changed places. What is the time now exactly?

10. A clock loses 10 seconds per hour and is set right at 9.15 a.m. on Monday. What will be the correct time when its hour and minute hands point exactly opposite between 9 and 10 p.m.? How must the hands be altered at noon on Tuesday so that the clock may show correct time?

MOTION.—A body moving at a steady speed covers equal distances in equal times. Its speed is the distance which it moves in one unit of time. The latter may be taken as a second, a minute or an hour. A body travelling 40 ft. per second covers 80 ft. in 2 sec., 120 ft. in 3 sec., and so on. Thus the distances covered in 1, 2, 3, . . . sec. form an A.P. 40, 80, 120 . . . . And the distance divided by the time is always the same: $\frac{40}{1} = \frac{80}{2} = \frac{120}{3}$. We call this ratio the speed, s, and this gives us a formula in which t = time in seconds, and d = distance in feet. Simple problems of steady motion can be solved by this formula. The units, of course, may be feet, yards, miles, seconds, minutes, etc., so long as we stick to one or the other.

$$s = \frac{d}{t} \quad \text{or} \quad d = st \quad \text{or} \quad t = \frac{d}{s}$$

Example

Two trains set out towards each other simultaneously from different stations 180 *miles apart. One train travels at* 45 *m.p.h., the other at* 60 *m.p.h. When and where do they meet?*

The important point in problems of this type is that *the time is the same for both.*

Let $d =$ distance of meeting-point from one station.

$\therefore$ $(180 - d)$ is the distance from the other station.

One train travels a distance d in time t at a speed of 45 m.p.h.

$$\therefore \quad \frac{d}{t} = 45 \qquad \therefore \quad d = 45t$$

The other train travels $(180 - d)$ miles at a speed of 60 in time t.

$$\therefore \quad \frac{180 - d}{t} = 60$$

From the preceding equation $d = 45t$, we have

$$\frac{180 - 45t}{t} = 60 \qquad \therefore \quad 180 - 45t = 60t$$

$$\therefore \quad 105t = 180 \qquad \therefore \quad t = \frac{12}{7} = 1\frac{5}{7}$$

Since

$$d = 45t$$

$$d = 45 \times \frac{12}{7} = 77\tfrac{1}{7}$$

Thus the time is $1\frac{5}{7}$ hours later and the distance $77\frac{1}{7}$ miles from the first station.

* * * * *

Problems of flow are very similar to problems of motion of bodies. If water flows along a pipe at the rate of 8 gallons a minute 8×12 gallons will have passed in 12 minutes. Thus if v is the total volume of water, t is the time and r is the rate of flow, the formula is:

$$r = \frac{v}{t} \quad \text{or} \quad t = \frac{v}{r} \quad \text{or} \quad v = rt$$

When the motion of one body is measured in relation to that of a body which is itself in motion we have to add or subtract the speed of the second body to get the correct speed of the first. Thus an aeroplane flying at 160 m.p.h. against a head wind of 50 m.p.h. is making a speed of only 110 m.p.h. as measured from the ground. Similarly a ship which can travel at 25 m.p.h. in still water would make 32 m.p.h. (or *knots*) if moving with a current of 7 m.p.h.

Ex. 13.42. Motion Problems

1. A walker sets off from Barchester at $3\frac{1}{2}$ miles an hour. A cyclist sets off half an hour later at 10 m.p.h. from Dorborough which is 40 miles away. When and where do they meet?
2. An express train travelling at 60 m.p.h. passes through a station just as a goods train travelling at 24 m.p.h. passes in the same direction through another station 6 miles ahead. In how many minutes will the express overtake the other, and how far from the second station?
3. A motor-boat which can travel 21 m.p.h. through still water goes on a journey up-river and back again. The journey up-stream takes twice as long as the journey down-stream. How fast is the river flowing?
4. A cyclist rides at 8 miles an hour from Ayton to Exton. He returns, with the wind behind him, at 12 m.p.h. The whole journey there and back took 5 hours. How long did the journey there take him, and how many miles was it?
5. There are 900 people at an exhibition and two turnstiles for letting them out. If one is stiffer than the other and lets 40 people per minute less than the other pass through, how long would the stiffer one take to pass all 900 if both together can do it in 10 minutes?
6. An aeroplane travelling at 242 m.p.h. passes a train moving in the same direction at 60 m.p.h. If the aeroplane takes $1\frac{1}{4}$ secs. to pass the train, how long is the train?
7. Three wheels revolve at different rates, the first doing 40 revolutions in one minute, the second 52 revolutions in 3 minutes and the third 60 revolutions in 7 minutes. Find the least time in which all will have made exact numbers of revolutions and the numbers of revolutions made by each in that time.
8. Tom can beat Dick by 50 yards in a race of 1,000 yards. Dick can beat Harry by 100 yards in a race of 1,000 yards. By how far could Tom beat Harry in 1,000 yards' race?
9. Guns are being fired at the rate of one every 2 minutes. A man travelling towards them hears the tenth explosion 17 minutes 50 seconds after the first. If sound travels at 1,100 feet per second, how fast is the man travelling?
10. Two equal boats set off towards each other from two points on a river 18 miles apart. The current is $1\frac{1}{2}$ m.p.h. and the boats can do 4 m.p.h. in still water. When and where will they meet?

Additional Exercises

Grateful acknowledgment is made to the following for permission to reproduce questions from recent School Certificate papers:

The Central Welsh Board (exercises marked C.W.B.).
The Northern Universities Joint Matriculation Board (exercises marked N.U.S.C.).
The Oxford Local Examinations Delegacy (exercises marked O.S.C.).
The Oxford and Cambridge Schools Examination Board (exercises marked O.C.S.C.).
The University of London (exercises marked L.G.S.).

The numbering runs on from that of the corresponding exercises in the text, but is not consecutive, as the exercises given in the text will in some cases provide all the practice likely to be needed.

Ex. 9.31

Make fully labelled charts similar to Chart 57 for each of the following equations. Check the graphical solution by algebra.

6. $2y = 4x + 7$
7. $3y = 6x + 4$
8. $2y = -4x + 3$
9. $3y = -3x - 4$
10. $2y = -5x + 6$
11. $2y = -5x - 3$
12. $y = +2{\cdot}5x + 1{\cdot}5$
13. $4y = 3x - 7$
14. $5y = 2x - 6$
15. $y = \frac{1}{2}x - \frac{3}{4}$

Ex. 9.32

Solve the following equations by graph, reducing—where necessary—to standard form. Check solutions by algebra.

6. $\dfrac{x}{3} - \dfrac{y}{4} = 1$
7. $\dfrac{3x}{2} + \dfrac{2y}{3} = 1$
8. $0{\cdot}5y + 0{\cdot}4x = 2$
9. $\frac{1}{2}(y - 3x) = 1$
10. $5x = 2(y - 1)$
11. $\frac{1}{2}(x - y) = 2(\frac{3}{2} - x)$
12. $\dfrac{y - 1}{2} = \dfrac{3 - x}{3}$
13. $\dfrac{y - 1}{x} = 4$
14. $\dfrac{x - 2}{y} = 4$
15. $\dfrac{3y}{7} = \frac{5}{14}(1 - x) + \frac{9}{14}(x - 1)$

Ex. 9.41

Find the simultaneous solutions of the following pairs of equations. (Reduce to standard form where necessary. Check by substitution.)

11. $x - 4y + 4 = 0$

    $\dfrac{x - 2y}{4} + x = 4$

12. $\dfrac{12x - y}{12} + \dfrac{2x - 3}{6} = 0$

    $\dfrac{4x}{15} + \dfrac{y}{30} = \frac{1}{5}$

13. $y = \dfrac{9x}{5} + 33$

    $y = \dfrac{-2x}{5}$

14. $2x + 3y = 15$

    $\dfrac{x}{5} + \dfrac{y}{10} = \frac{5}{6}$

15. $x + 2y = 1$

    $\dfrac{3(x - y)}{10} = -y$

16. $2(2x - 3y + 2) = x + y - 6$

    $4(x + y - 7) = 3(x - y + 2)$

17. $5(3x - 4y + 2) = 2(7x - 3y - 2)$

    $\dfrac{3x + 4y}{5} + x = \dfrac{2x + 5y + 3}{10}$

18. $\dfrac{x}{4} + \dfrac{y}{5} = 6$

    $\dfrac{x}{2} + \dfrac{y}{3} + 8 = \dfrac{x}{3} + y$

19. $3x - 2y + 1 = 0$

    $\dfrac{x + y}{2} = \dfrac{3x - 8y}{5}$

20. $5y - 2x = 3$

    $y = 15x - 14$

Ex. 9.52

11. Solve the equations (graphically or otherwise):

$$4x - 3y = -8; \quad 3x - \frac{y}{4} = -2 \quad \text{(C.W.B.)}$$

12. Solve the equations:

$$\frac{2y}{3} - \frac{3 - 4x}{7} = 0; \quad 4x = 7 - 2y \quad \text{(C.W.B.)}$$

13. Solve: $9x + 6y = 4x - 9y = 14$ (L.G.S.)

14. Solve: $y = \dfrac{5(x + 4)}{8}; \quad x = \dfrac{y + 3}{2}$

15. Solve: $\dfrac{5(3x - 4y)}{4} = 3(x - 2y) = 5(3x + y - 3)$

16. Solve: $2x + y = 3; \quad x + 2y = 4$ (N.U.S.C.)

17. Solve: $4x + 30y = \frac{4}{3}(x - 3y); \quad 14x - 54y + 155 = 0$

18. Solve: $x + 17y = 21 - x - 2y; \quad 19(3x - y) = 539\frac{1}{2}$

19. Solve: $37{\cdot}5x - 15y = 7{\cdot}5; \quad 0{\cdot}14x + 0{\cdot}07y = 0{\cdot}91$

20. Solve: $\frac{1}{6}(5x + 2y) = x + y = 2x + 3y + 1$

21. Solve: $\dfrac{2x + 3y}{5} = x = \dfrac{5x - y - 4}{2}$

22. Solve: $\dfrac{7x - 4y - 12}{3} = 1 = \dfrac{9x - 6y - 13}{5}$

23. Solve: $9x - 4y = 5; \quad 11x - \dfrac{4y}{5} = 7 + 2x + 3y$

24. Solve: $\dfrac{6x + 3y - 1}{3} = \dfrac{3x + 3y + 2}{5}; \quad \dfrac{3(5x + 4y)}{13} = 1$

25. Solve: $\dfrac{10 + 5x - 4y}{49} - \dfrac{3y + x}{35} = 1; \quad 12x + 15\frac{1}{2}y = 1\frac{1}{2}$

Ex. 9.53

Solve the following:

11. $3x + y + z = 0$
    $x + y + z = 2$
    $x + 3y + 2z = 6$

12. $x + y + z = 10$
    $x + 4y + 2z = 21$
    $2x + 3y + 3z = 27$

13. $2x + 3y + 2z = -2$
    $x + \dfrac{y}{3} - \dfrac{5z}{3} = 4\frac{2}{3}$
    $5x - 6y - 6z = 1$

14. $2x + 3y + 2z = -2$
    $3x + y - 5z = 14$
    $5(x - y) = 1 + y + 6z$

15. $x - 1\frac{1}{2} + \dfrac{y + z}{2} = -11$
    $x + z + 2(y + 10) = 0$
    $x + 7 = -(y + 2)$

16. $3x + 2y + 5z = 11$
    $x - 3y + z = 9$
    $2x + 3y + 3z = 3$

17. $\dfrac{2x - y + z}{2} = 2$
    $\dfrac{3x + y}{2} = 3 + \dfrac{x - y}{2}$
    $3x + 2y + 2z = 10$

18. $\dfrac{3x + z}{3} = \dfrac{2z + y + 2}{5}$
    $x + y + z = 6$
    $3(x - y) + 2 = y - z$

19. $x + 3y + 2z = 4$
    $\dfrac{2x - y - z}{2} = x + 3y + z$
    $3x - y - z = 12$

20. $x + 2y + 4z = 1$
    $4(x + y) = \dfrac{20z + 5x - 4y}{6}$
    $\dfrac{3x - 4y}{5} + 4 = 5x + 4y + 8z$

Ex. 10.11

Write down the values of the constants F_0, a and b for:

4. M_n in Chart 9.
5. V_n in Chart 9.
6. H_n in Chart 10.
7. F_n in Chart 11.
8. X_n in Chart 24.
9. C'_n in Chart 4.
10. M_n in Chart 114.
11. ${}_8S_n$ in Chart 115.
12. T_{n-2} in Chart 111.
13. ${}_nB_{n-1}$ in Chart 6.
14. P_n in Chart 20.
15. H_n in Chart 20.

Ex. 10.12

Construct numerical series from $n = -5$ to $n = +5$ for the following quadratic functions. Make histograms based thereon, and construct a difference table like the above for each. Choose a vertical scale suitable for each.

6. $F_n = n^2 + 2n + 1$
7. $F_n = n^2 - 3n + 2$
8. $F_n = 5n^2 + 4n$
9. $F_n = 7n^2 - 10n + 4$
10. $F_n = 2n^2 - 2n + 1$
11. $F_n = (n - 3)(n - 4)$
12. $F_n = (n - 4)(n + 1)$
13. $F_n = \frac{1}{2}(n^2 - 3n + 2)$
14. $F_n = 8n^2 - 8n + 1$
15. $F_n = (2n - 1)(n + 2)$

Ex. 10.23

Plot the parabola $y = x^2$ with the following points for origin and write the new equation for each in both the familiar and the disguised form:

6. $(4, 0)$
7. $(3, 4)$
8. $(1\frac{1}{2}, 2)$
9. $(-2\frac{1}{2}, -5)$
10. $(3{\cdot}6, 4{\cdot}2)$

Ex. 10.24

Change the following equations to the familiar form and so locate the vertex. Then plot the graph.

6. $y = x^2 + 4x + 1$
7. $y = x^2 - 8x + 19$
8. $y = x^2 - x + \frac{5}{4}$
9. $y = x^2 + 2x$
10. $y = x^2 - x + 1{\cdot}75$
11. $2y = 2x^2 + 4x + 2$
12. $y = x^2 + 6x + 10$
13. $y = x^2 - 12x + 38$
14. $y = x^2 - 10x + 20$
15. $y = x^2 - \frac{x}{2} + \frac{3}{4}$

Ex. 10.31

Plot the following quadratics by changing scale and origin and using the stencil:

6. $4x^2 + x - 3$
7. $3x^2 - x - 2$
8. $2x^2 + x + 2$
9. $5x^2 + 13x + 6$
10. $2x^2 + 7x + 6$
11. $7x^2 + 10x - 8$
12. $3x^2 + 7x + 4$
13. $2x^2 + x - 3$
14. $4x^2 + 15x - 4$
15. $3x^2 + 11x + 6$

Ex. 10.33

Test the following for real roots by the formula. Plot graphs by stencil. Where roots are real, find them from the graph. Check by formula.

11. $10x^2 - 11x + 1$
12. $15x^2 + 12x + 5$
13. $15x^2 + 28x - 32$
14. $19x^2 + 11x + 17$
15. $10x^2 + 5x + 1$
16. $18x^2 - 57x + 35$
17. $20x^2 + 51x + 27$
18. $20x^2 + 11x + 5$
19. $15x^2 + 7x + 6$
20. $9x^2 + 6x + 7$

Ex. 10.41

Draw graphs similar to Chart 64 of the following quadratics, starting with the three values shown. Plot 10 points for each parabola.

6. $y = \frac{1}{4}x^2$ $\quad x = 0, 2, 4$
7. $y = \frac{1}{3}x^2$ $\quad x = 0, -3, -6$
8. $y = \dfrac{x^2}{2}$ $\quad x = 0, -2, -4$
9. $y = \frac{1}{4}x^2 - 3x + 8$ $\quad x = 0, 2, 4$
10. $y = 10x^2 - 7x + 2$ $\quad x = 0, \frac{1}{10}, \frac{2}{10}$

Ex. 10.51

Draw diagrams like the one in Chart 65 for:

6. $x^2 + 8x - 20$
7. $x^2 + 12x + 32$
8. $x^2 + 2x - 8$
9. $x^2 + 5x - 12\frac{1}{2}$
10. $x^2 + 9x + 8$
11. $x^2 + x - 2$
12. $x^2 + 13x + 36$
13. $x^2 + 11x + 10$
14. $x^2 + 15x - 34$

Ex. 10.52

Draw rectangles as in Chart 66 to represent the following areas. Write the value of the net area over each rectangle.

11. $(+4-5) \times (+3-2)$
12. $(+3-6) \times (-2-3)$
13. $(+5-3-4) \times (+3-2)$
14. $(-7+5) \times (-3+1)$
15. $(-6+4-3) \times (+4-5+2)$
16. $(-9+7) \times (-7+9)$
17. $(-9+7-2) \times (-4-3+2)$
18. $(+6+2-3) \times (-1+2)$
19. $(-6-2+7) \times (-4+2-3)$
20. $(7+2-4) \times (-4-1+3)$

Ex. 10.62

Repeat the processes carried out in examples 1–4 for the following:

6. Give s the values 7, 8, 9, 10 in the formula

$$_sF_n = 1 + (s-1)(n-1) + (s-2) \,.\, T_{n-2}$$

7. Give s the values 7, 8, 9, 10 in the formula

$$_sM_n = 1 + s \,.\, T_{n-1}$$

Ex. 10.71

Obtain the simultaneous solutions of $y = x^2$ with the following linear equations:

11. $y = 4$
12. $y = 9$
13. $y = 2x + 3$
14. $y = \dfrac{4x}{3} - \frac{1}{3}$
15. $y = \dfrac{5x}{4} + \frac{3}{2}$
16. $y = -\dfrac{4x}{3} + \frac{4}{3}$
17. $y = -5x$
18. $y = -3x - 2$
19. $5y + 12x + 4 = 0$
20. $8y + 2x - 3 = 0$

Ex. 10.73

11. Draw the graph of $\frac{1}{4}x(13-2x)$ between $x = 0$ and $x = 6$, taking the side of one large square along each axis to represent one unit.
    Find from your graph, as accurately as it will allow, the roots of the equation $x(13-2x) = 19$. (O.S.C.)
12. Draw the graph of $y = 3x - x^2$ plotting points at half-unit intervals of x from $x = -0{\cdot}5$ to $x = 3{\cdot}5$. (Take one inch for unit on both axes.)
    From your graph find:
    (i) The roots of the equation $3x - x^2 - 1 = 0$.
    (ii) The range of values of x for which $3x - x^2$ is greater than $\frac{1}{2}$. (N.U.S.C.)
13. With the same scales and axis, draw the graphs from $x = -2$ to $x = +2$ of $y = 2x^2$ and $4y = 6x + 9$.
    Use your graphs to find:
    (i) The value of $\sqrt{1{\cdot}6}$.
    (ii) The values of x between which the expression $6x + 9 - 8x^2$ is positive. (C.W.B.)

14. Draw the graph of $\frac{1}{2}(x^2 - 2x - 2)$ from $x = -1$ to $x = +5$, taking 1 in. to represent a unit on each axis. Use your graph to solve the equations

    (i) $x^2 - 2x - 2 = 0$

    (ii) $2x^2 - 7x - 3 = 0$

    and explain your method clearly.

15. Draw the graph of $y = 8 + 3x - x^2$ for values of x from -2 to $+4$ taking 1 in. to represent one unit of x and $\frac{1}{2}$ in. to represent one unit of y.

    Use your graph to determine:

    (i) The roots of the equation $3x - x^2 = -3$.

    (ii) The value of K for which the equation $8 + 3x - x^2 = K$ has equal roots. (L.G.S.)

16. Taking 1 in. (or 2 cm. if you use centimetre paper) as unit on each axis, draw the graph of

$$y = x^2 - 5x + 1$$

    for values of x from 0 to 5.

    Use the graph to find approximately the roots of the equation

$$x^2 - 5x + 2 = 0 \quad \text{(O.C.S.C.)}$$

17. Draw in the same diagram the graphs of $y = x^2 - 2x - 2$ and $y = \frac{1}{2}(x - 3)$ from $x = -1$ to $x = 3$, taking 1 in. for unit in both axes and plotting the values of y corresponding to half-unit intervals of x.

    Show that the x-co-ordinates of the intersections of the graphs are the roots of the equation $2x^2 - 5x - 1 = 0$, and from the graphs give the roots as accurately as possible. (N.U.S.C.)

18. Draw the graph of

$$y = \tfrac{1}{8}(x + 2)(2x - 3)$$

    between the values $x = -4$ and $x = 3$, taking the side of one large square to represent one unit along each axis.

    Use your graph to find the roots of the equation

$$(x + 2)(2x - 3) = 2\tfrac{1}{2} \quad \text{(O.S.C.)}$$

19. Draw the graph of

$$y = (5x + 1)(7 - x)$$

    between $x = 0$ and $x = 7$, taking the side of one large square to represent one unit along the axis of x and ten units along the axis of y.

    Use your graph to find:

    (i) The roots of the equation $(5x + 1)(7 - x) = 45$.

    (ii) The range of values of x for which $(5x + 1)(7 - x)$ is greater than 57. (O.S.C.)

20. Draw the graph of the function

$$y = \tfrac{1}{2}(x^2 - 3x - 3)$$

from $x = -2$ to $x = +4$.

Draw also the graph of $y = \tfrac{1}{2}(2 - x)$ and find from the graphs the roots of the equations

(i) $x^2 - 3x - 3 = 0$

(ii) $x^2 - 2x - 5 = 0$ (N.U.S.C.)

21. Using 1 in. as unit for both x and y and the same axes, draw the graphs from $x = -4$ to $x = +4$ of $y = \tfrac{1}{2}x^2$ and of $2x = 9 - 4y$.

Use one of the graphs to find the value of $\sqrt{2 \cdot 5}$.

What quadratic equation has as its solutions the x-value of the intersections of your graphs. (C.W.B.)

Ex. 11.12

Solve graphically the following:

6. $x^3 + 3x^2 = x + 3$
7. $x^3 - 2x^2 - 5x + 6 = 0$
8. $x^3 + 4x^2 + x = 6$
9. $x^3 + 12 = 3x^2 + 4x$
10. $x^3 - 2x^2 - 9x + 18 = 0$
11. $x^3 + 3 = 3x^2 + x$
12. $x^3 + 3x^2 - 4x - 12 = 0$
13. $x^3 + 16 = x^2 + 16x$
14. $x^3 - 6x^2 + 5x + 12 = 0$
15. $2x^3 - x^2 - 7x + 6 = 0$

Ex. 11.21

Draw rectangles to represent the following quadratics for each of the five x-values shown.

9. $y = (x + 4)(x + 5)$ $\quad x = -7, -6, -5, -4, -3$
10. $y = (x - 6)(x - 7)$ $\quad x = 5, 6, 7, 8, 9$
11. $y = (x - 1)(x + 1)$ $\quad x = -2, -1, 0, 1, 2$
12. $y = (x - 2)(x + 2)$ $\quad x = -3, -2, 0, 2, 3$
13. $y = (x + 4)(x + 4)$ $\quad x = -6, -5, -4, -3, -2$
14. $y = x(x - 3)$ $\quad x = -1, 0, 1, 3, 5$
15. $y = (x - 6)(x + 2)$ $\quad x = -3, -2, +2, +6, +8$

Ex. 11.22

Factorize the following quadratics. Plot their graphs. Show that the roots correspond with the factors. Draw rectangles for the root-values of x and so show that the algebraic areas of these rectangles are all zero.

11. $y = x^2 + x - 2$
12. $y = x^2 + 2x - 8$
13. $y = x^2 + 8x + 15$
14. $y = x^2 + 2x - 24$
15. $y = x^2 - 3x - 18$
16. $y = x^2 - 16$
17. $y = x^2 + 5x - 24$
18. $y = x^2 - 7x - 18$
19. $y = x^2 - 6x + 8$
20. $y = x^2 + 2x - 15$

Ex. 11.26

Make charts similar to Chart 72, adding the ordinates graphically, for the following:

3. $x^3 - 13x^2 - 10x + 400$
4. $x^3 - 19x^2 + 40x + 528$
5. $x^3 - 7x^2 - 14x + 120$
6. $x^3 - 13x^2 + 26x + 112$
7. $x^3 - 15x^2 + 14x + 360$
8. $x^3 - 9x^2 - 36x + 324$
9. $x^3 - 7x^2 - 53x + 315$
10. $x^3 - 20x^2 + 116x - 160$

Ex. 11.31

Find by means of a cubic and quadratic graph the *real* roots of the following equations:

9. $x^3 + x^2 + 6x - 24 = 0$
10. $x^3 + 8x^2 + 45x + 90 = 0$
11. $x^3 - 6x^2 + 13x - 8 = 0$
12. $x^3 + 32x - 33 = 10x^2$
13. $x^3 - 4x^2 = 2x + 15$
14. $x^3 - 2x^2 - x = 28$
15. $x^3 + 2x^2 + x + 12 = 0$

Ex. 11.32

9–15. Solve each of the equations of examples 9–15 in Ex. 11.31 by:

(a) reducing to the form $X^3 + pX + q = 0$;
(b) finding where the straight-line graph $y = -pX - q$ cuts the cubic curve $y = X^3$.

16. Draw the graph of $y = x(x^2 - 16)$ from $x = -4$ to $x = 4$, taking the side of one large square to represent one unit along the axis of x and ten units along the axis of y.

Use your graph to find:

(i) Two values of x, in the given range, which satisfy the equation

$$x^3 - 16x - 10 = 0$$

(ii) The value of x for which $x(x^2 - 16)$ has the least value in the given range. (O.S.C.)

17. Draw a graph of

$$y = \tfrac{1}{8}(50x - x^3)$$

between the values of $x = 0$ and $x = 7$, taking the side of one large square to represent one unit along the axis of x and ten units along the axis of y.

Use your graph to find two values of x between 0 and 7, each of which satisfies the equation

$$x^3 - 50x + 80 = 0 \quad \text{(O.S.C.)}$$

18. Draw the graph of

$$y = x(8 - x)^2$$

between the values $x = 0$ and $x = 8$, taking the side of one large square to represent one unit along the axis of x and ten units along the axis of y.

Use your graph to find two values of x between 0 and 8, each of which satisfies the equation

$$x(8 - x)^2 = 60 \quad \text{(O.S.C.)}$$

Ex. 11.41

Solve, using the method shown in § 5,

7. $x^3 - 3x^2 - x + 3 = 0$
8. $x^3 + 6x^2 + 3x - 10 = 0$
9. $x^3 - 15x^2 + 59x = 45$
10. $x^3 + 6x^2 - 13x = 42$
11. $x^3 + 9x^2 - 37x = 165$
12. $8x^3 - 12x^2 - 386x + 195 = 0$
13. $64x^3 + 48x^2 - 244x - 63 = 0$
14. $8x^3 + 12x^2 - 66x = 35$
15. $x^3 - 30x^2 + 275x - 750 = 0$

Ex. 11.42

Find one real root of each of the following equations:

9. $x^3 + x = 2$
10. $x^3 - 4x - 15 = 0$
11. $x^3 - 10x + 24 = 0$
12. $x^3 + 4x + 16 = 0$
13. $x^3 - 36x = 91$
14. $x^3 + 3x - 140 = 0$
15. $x^3 + 6x + 560 = 0$
16. $x^3 + 4x + 5 = 0$
17. $x^3 + 4x + 240 = 0$
18. $x^3 - 18x = -35$
19. $x^3 - 11x = 20$
20. $x^3 - 74x = 260$

Ex. 12.11

Draw histograms like Chart 79 for:

6. $n(H_n - 4) = 5$
7. $n(H_n + 4) = 5$
8. $n(H_n + 5) = -6$
9. $2n(H_n - 6) = 8$
10. $H_n - 3 = \dfrac{4}{n-2}$
11. $H_n = \dfrac{5}{n-3} + 4$
12. $\dfrac{4}{n-4} = H_n + 2$
13. $\dfrac{H_n}{5} = \dfrac{1}{n-2} + 1$
14. $\dfrac{H_n - 2}{4} = \dfrac{1}{n-3}$
15. $(H_n - 4)(n - 4) = 1$

Ex. 12.12

With the same scale for x and y draw the graphs of the continuous function corresponding to those represented by the histograms of Ex. 12.11, i.e. to

6. $x(y - 4) = 5$
7. $x(y + 4) = 5$
8. $x(y + 5) = -6$
9. $2x(y - 6) = 8$
10. $y - 3 = \dfrac{4}{x-2}$
11. $y = \dfrac{5}{x-3} + 4$
12. $\dfrac{4}{x-4} = y + 2$
13. $\dfrac{y}{5} = \dfrac{1}{x-2} + 1$
14. $\dfrac{y-2}{4} = \dfrac{1}{x-3}$
15. $(y - 4)(x - 4) = 1$

Ex. 12.22

Plot the following. Find the origin with reference to which the hyperbola is rectangular by inspection of the asymptotes and by algebraic substitution as above:

5. $y = \dfrac{3x + 5}{2x + 3}$
6. $y = \dfrac{4x - 9}{3x + 2}$
7. $y = \dfrac{15x - 4}{5x - 2}$
8. $y = \dfrac{12x + 15}{4x + 3}$
9. $y = \dfrac{7x + 4}{3x + 5}$
10. $y = \dfrac{11x + 8}{4x + 1}$

Ex. 12.23

Plot each of the graphs in Ex. 12.12 on the same grid with four scales:

(*a*) 1 s.p.u. for $x = 1 = y$;
(*b*) 1 s.p.u. for $x = 1$, 3 s.p.u. for $y = 1$;
(*c*) 1 s.p.u. for $y = 1$, 3 s.p.u. for $x = 1$;
(*d*) 3 s.p.u. for $x = 1 = y$.

Ex. 12.31

Draw diagrams like those of Chart 82 for rectangles of area:

4. 4 sq. cm.
5. 49 sq. cm.
6. $6\frac{1}{4}$ sq. cm.
7. 64 sq. cm.
8. $12\frac{1}{4}$ sq. cm.
9. 30·25 sq. cm.
10. 20·25 sq. cm.

Ex. 12.32

Plot the following:

4. $y = \dfrac{3 - 2x}{x - 3}$
5. $y = \dfrac{4 - 5x}{x - 4}$
6. $y = \dfrac{3 - 2x}{x - 2}$
7. $y = \dfrac{1 - 3x}{x - 3}$
8. $y = \dfrac{5 - 6x}{x - 2}$
9. $y = \dfrac{6 - 7x}{x - 5}$
10. $y = \dfrac{2 - 3x}{x - 3}$

Ex. 12.41

Use the method of the last two examples to test which of the following are harmonic series:

6. $\frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{6}{11}, \frac{1}{2}$
7. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$
8. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}$
9. $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \frac{2}{15}$
10. $\frac{4}{15}, \frac{2}{7}, \frac{1}{3}, \frac{4}{11}, \frac{2}{5}$
11. $\frac{12}{7}, \frac{3}{2}, \frac{4}{3}, \frac{6}{5}$
12. $\frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \frac{2}{3}$
13. $1\frac{2}{5}, 1\frac{1}{6}, \frac{7}{8}, \frac{7}{9}$
14. $3, 2\frac{2}{3}, 2\frac{2}{5}, 2\frac{2}{11}$
15. $1\frac{3}{4}, 1\frac{1}{6}, \frac{7}{8}, \frac{7}{10}$
16. $\frac{9}{10}, \frac{9}{11}, \frac{9}{13}, \frac{3}{5}$
17. $6\frac{1}{4}, 4\frac{1}{6}, 3\frac{1}{8}, 2\frac{1}{2}$
18. $4, 3{\cdot}6, 3, 2\frac{4}{7}$
19. $8, 6\frac{6}{7}, 6, 5\frac{1}{3}$
20. $7{\cdot}2, 6, 5\frac{1}{7}, 4\frac{1}{2}, 4$

Ex. 12.42

Find the harmonic mean of the following numbers:

7. 8 and 12
8. 14 and 26
9. 7 and 23
10. -5 and 9
11. -9 and 15
12. -10 and -14
13. $3\frac{1}{2}$ and $10\frac{1}{2}$
14. 19 and -4
15. $3\frac{3}{4}$ and $11\frac{1}{4}$

Ex. 12.43

Find a formula for the following harmonic series:

6. $\frac{2}{3}, \frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \frac{2}{15}$
7. $\frac{1}{5}, \frac{2}{15}, \frac{1}{10}, \frac{2}{25}, \frac{1}{15}$
8. $4\frac{4}{15}, 4, 3\frac{13}{17}, 3\frac{5}{9}, 3\frac{7}{19}$
9. $1\frac{8}{17}, 1\frac{6}{19}, 1\frac{4}{21}, 1\frac{2}{23}, 1$
10. $1\frac{1}{11}, 1\frac{1}{5}, 1\frac{1}{3}, 1\frac{1}{2}, 1\frac{5}{7}$
11. $1\frac{1}{6}, 1\frac{3}{11}, 1\frac{2}{5}, 1\frac{5}{9}, 1\frac{3}{4}$
12. $5, 3, 2\frac{1}{7}, 1\frac{2}{3}, 1\frac{4}{11}$
13. $1\frac{1}{5}, 1\frac{1}{3}, 1\frac{1}{2}, 1\frac{5}{7}, 2$
14. $21, 6, 3\frac{1}{2}, 2\frac{8}{17}, 1\frac{10}{11}$
15. $3\frac{3}{5}, 3\frac{3}{11}, 3, 2\frac{10}{13}, 2\frac{4}{7}$

Ex. 12.44

Test the Δ formula by recourse to the harmonic series of Ex. 12.43.

Ex. 13.11

Write down the degree m and coefficients $a, b, c \ldots$ of the following polynomials:

11. $y = x^5 + 4x^3 + 2x + 5$
12. $y = 3x^6 + 4x + 7$
13. $2y = 4x^3 + 6x^2 + 8x + 3$
14. $3y = 9x^3 + 4x + 2$
15. $5y = 5x^4 + 10x^3 + 15x + 2$
16. $y = x^7 - 1$
17. $2y = 4x^4 - 6x^2 - 5$
18. $x + y = x^3 - 4x^2$
19. $2x^2 + 3y = x$
20. $3x^3 - 4x^2 + 1 = y - x$

Ex. 13.42. MOTION PROBLEMS

11. A boy cycles to school every morning in 45 minutes. On one occasion he sets out 10 minutes later than usual and, cycling 3 miles per hour faster than usual, reaches school at the normal time. What is his normal speed? (L.G.S.)
12. A man walks for h hours at a speed of x m.p.h. He then rides for 5 miles at a speed of y m.p.h. Find (*a*) the total distance travelled, (*b*) the total time taken, and (*c*) the average speed for the whole journey. (L.G.S.)
13. A train travels uniformly over 9 miles of track. If the speed were increased by 9 m.p.h., the time would be reduced by 3 minutes. What is the time for the 9 miles at the original speed? (C.W.B.)
14. A man has been accustomed to motor from his house to his office, a distance of 4 miles. Now he has to walk to a bus stop $\frac{1}{2}$ mile from his house and then board a bus which takes him to his office. Assuming that he boards the bus 10 minutes after leaving his house, the whole journey takes 15 minutes longer than by car. If the average speed of the bus is 10 m.p.h. less than the average speed of the car, find the average speed of the bus. (N.U.S.C.)
15. A train starts at the scheduled time over a journey of 90 miles, but arrives 20 minutes late. Find its average speed, if this is less by 3 m.p.h. than it would have been if the train had run to time.
16. On a journey of 90 miles a train is 10 minutes late; if its average speed had been increased by 5 m.p.h. it would have been 5 minutes early. Find the actual average speed of the train on the journey and deduce the average speed when the train is running to time. (C.W.B.)
17. The cruising speeds of two aeroplanes are such that one is 60 m.p.h. more than the other. The slower plane takes 10 seconds longer to fly a mile in calm air at cruising speed than the faster plane. Find the cruising speed of each plane and the time each takes to fly the mile. (L.G.S.)
18. A car makes a journey of 144 miles, stopping for 1 hour by the way. Had it travelled at an average speed of 4 m.p.h. faster and stopped for $1\frac{1}{2}$ hours, it would have taken the same time. What is its average speed? (The time taken over the stops is not included when calculating average speeds.) (N.U.S.C.)
19. A train runs "non-stop" daily from X to Y, a distance of 140 miles. On a busy day more coaches are added, and its average speed is thus reduced by 10 m.p.h., so that it is 20 minutes late in reaching Y. Find its usual average speed and verify your answer by calculating the actual times of the two journeys. (N.U.S.C.)
20. Two cars set out over the same run, the faster starting half an hour after the slower. The slower travels at a speed of 30 m.p.h., the faster at a speed of x m.p.h.

    Show that the faster catches up the slower in $\dfrac{15}{x-30}$ hours.

    Two cars A and B start together and travel along the same run at speeds of 30 and 35 m.p.h. respectively. A faster car C starts half an hour after them, travels at a uniform speed, and passes B 25 minutes after it has passed A. Find the speed of C. (O.S.C.)

Ex. 13.43. Miscellaneous Problems

1. The members of a club hire a charabanc for a trip, the cost being a certain fixed sum however many go on the trip. Assuming that all the members can go, the cost works out at 5s. 6d. each. Three members, however, are unable to go, and the rest have to pay 6s. 3d. each. Find the sum paid for the use of the charabanc. (O.S.C.)
2. A bookseller sold a cheap edition of a certain book at 2s. 0d. a copy, the ordinary edition at 3s. 6d. a copy, and an illustrated edition at 15s. 0d. a copy. He sold altogether 215 copies, receiving £46 0s. 6d., and noted that of the copies sold the number of illustrated copies was one-third of the number of copies of the cheap edition. How many copies of each edition did he sell? (N.U.S.C.)
3. A dealer spends £13 in buying two kinds of cloth, one at 2s. a yard, the other at 3s. 6d. a yard. By selling the cheaper at 2s. 6d. a yard and the dearer at 4s. 6d. a yard, he receives, in all, £16 10s. Find how many yards of the cheaper cloth he buys. (O.S.C.)
4. Find the two numbers which are such that one-quarter of their sum is 8 less than the larger one and one-third of their difference is 12 less than twice the smaller one. (O.C.S.C.)
5. The average age of a class of 10 boys is 15 years. The average age of these boys and their form master is 17 years. What is the age of the form master? (O.C.S.C.)
6. The ratio of two numbers is $3/2$. When each number is increased by $\frac{1}{2}$, the ratio becomes $23/16$. Find the numbers. (C.W.B.)
7. The difference of two numbers is 2. If each number is increased by 5, their product is increased by 105. What are the numbers? (C.W.B.)
8. A motorist, travelling from one town to another, takes 5 hours for the journey. His average speed over the first 80 miles is one and a half times his average speed over the rest of the journey, and is greater by 6 m.p.h. than his average speed over the whole journey. Find the distance between the two towns. (O.S.C.)
9. Find three consecutive odd numbers such that the sum of twice the smallest and three times the second exceeds four times the largest by 21. (N.U.S.C.)
10. A man made a will leaving £7,920 to be divided equally amongst his surviving relatives. Four of the relatives died before the man himself, and each survivor received £80 more than if all the relatives had lived. How many relatives actually benefited?

    If only two of the relatives had died instead of four, how much extra would each of the others have received instead of the £80? (L.G.S.)
11. A grocer bought a quantity of tea for £700 and a quantity of coffee for £1,120, the price of coffee being £2 per cwt. more than the price of the tea. The total weight of tea and coffee purchased being 6 tons, find the price per cwt. of the tea. (L.G.S.)
12. The sum of 2 numbers is $11\frac{3}{8}$, and the result of subtracting the smaller from the larger is 3 more than the result of dividing the larger by the smaller. Find the two possible values of the smaller number.

13. A number of cases contain, in all, 2,520 articles, each case containing the same number of articles. The articles are repacked so that now each case contains 1 less article than before, and 2 more cases are needed. Find the number of cases in the original packing.

14. The total cost of a number of pullets at a certain date was £18 5s. 6d. At a later date the price was reduced by 2s. 0d. each, 4 more than before were bought, and the total cost was then £20 9s. 6d. Find the original cost of each pullet. (N.U.S.C.)

15. A man agrees to sell the Government 2,000 tons of sand at 18s. per ton. He has some in stock for which he has paid 15s. per ton, but it is not enough, so he has to buy more at 22s. per ton. On the whole deal he makes £20. How many tons had he to buy? (N.U.S.C.)

Ex. 13.45. Problems Involving Arithmetical Progressions

1. The sum of 10 terms of an arithmetic progression is 285 and the 10th term is 60. Find the sum of 20 terms and the 20th term. (L.G.S.)

2. If the last term of an arithmetic progression of 21 terms is 117 and the middle term is 63, find the first term.
Find also the sum of all the terms following the tenth (C.W.B.)

3. The sum of the first 10 terms of an arithmetical progression is equal to three-fifths of the sum of the next 10 terms. If the 10th term is 78, find the 20th term. (O.S.C.)

4. The sum of the first 3 terms of an arithmetic progression is 114 and the 12th term is 8. Find the value of the smallest positive term of the series and the sum of the first 30 terms. (L.G.S.)

5. The fifth term of an arithmetical progression is 14 and the twentieth term is -31. Find the value of the first negative term and the sum of all the positive terms. (C.W.B.)

6. The first terms of two arithmetical progressions are equal, and the 7th term of the first progression is equal to the 10th term of the second. Find the ratio of their common differences. If the sum of the first 8 terms of the second progression is 136, find the first term. (O.S.C.)

7. The first term of an arithmetic progression is 3 and the sum of the first 10 terms is 60 less than the sum of the first 4 terms. Find the first 3 terms. (C.W.B.)

8. Find the first term and common difference of an arithmetical progression in which the 30th term is twice the 8th term, and the sum of the first 8 terms is 111. (O.S.C.)

9. The fifth term of an arithmetic progression is 9, and the sum of the first 15 terms is 360. Find the common difference and the first term. (L.G.S.)

10. The first term of an arithmetical progression is 3, the sixth term is 6, and the last term is 18. Find the common difference and the sum of all the terms. (O.S.C.)

11. The first and last terms of an arithmetical progression of n terms are -4 and 45 respectively. Write down an expression for the sum of these n terms. If this sum is $389\frac{1}{2}$, find the value of n and the common difference. (C.W.B.)

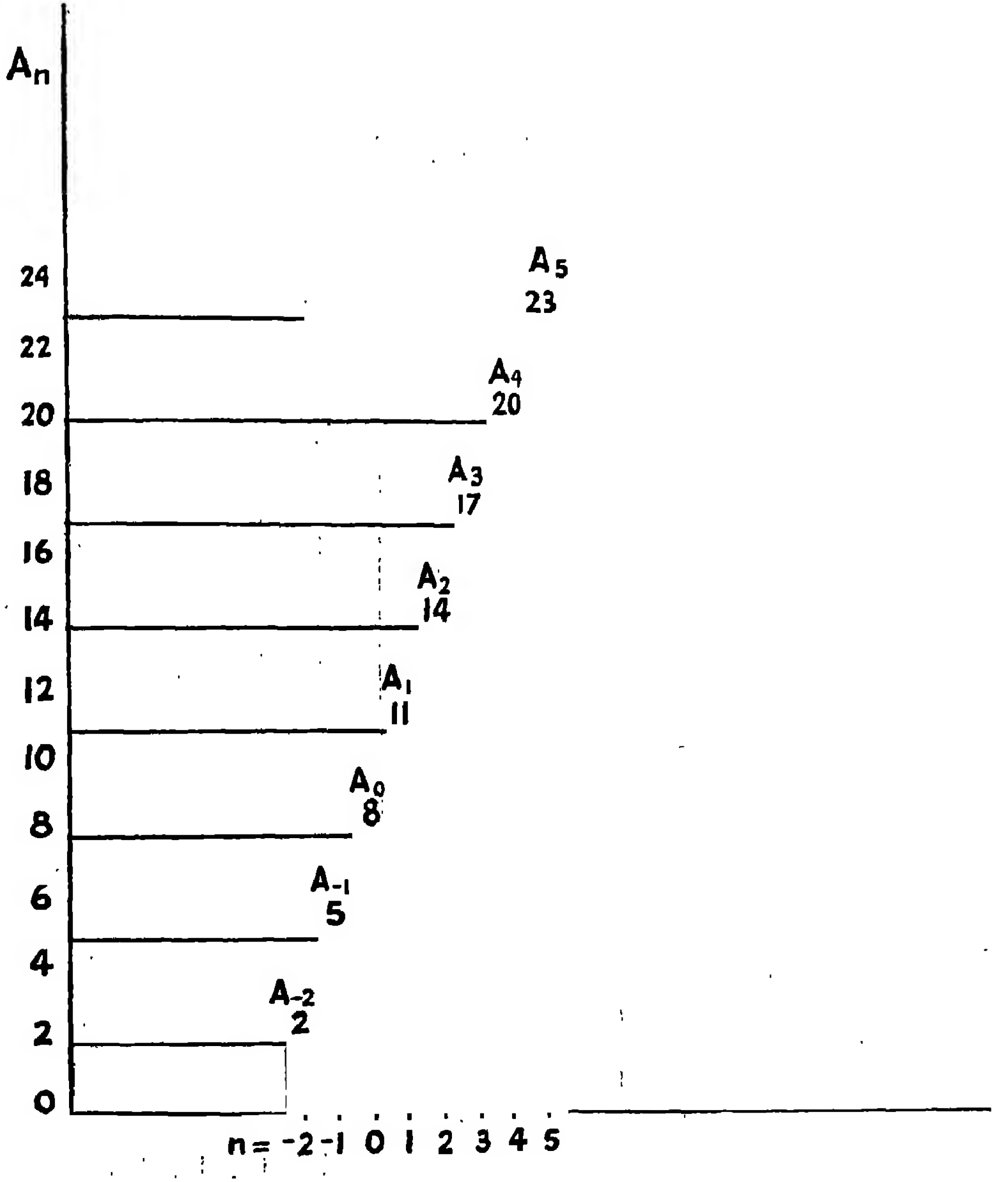
12. Find the sum of the numbers between 500 and 1,000 which are multiples of 3. (C.W.B.)

13. The sum of the first 3 terms of an arithmetic progression is 114 and the 12th term is 8. Find the value of the smallest positive term of the series and the sum of the first 30 terms. (L.G.S.)

14. The sum of the first n terms of an arithmetical progression is $\frac{n(2n+3)}{2}$. Find the first term, the common difference and the nth term. (L.G.S.)
(Hint: give n different values.)

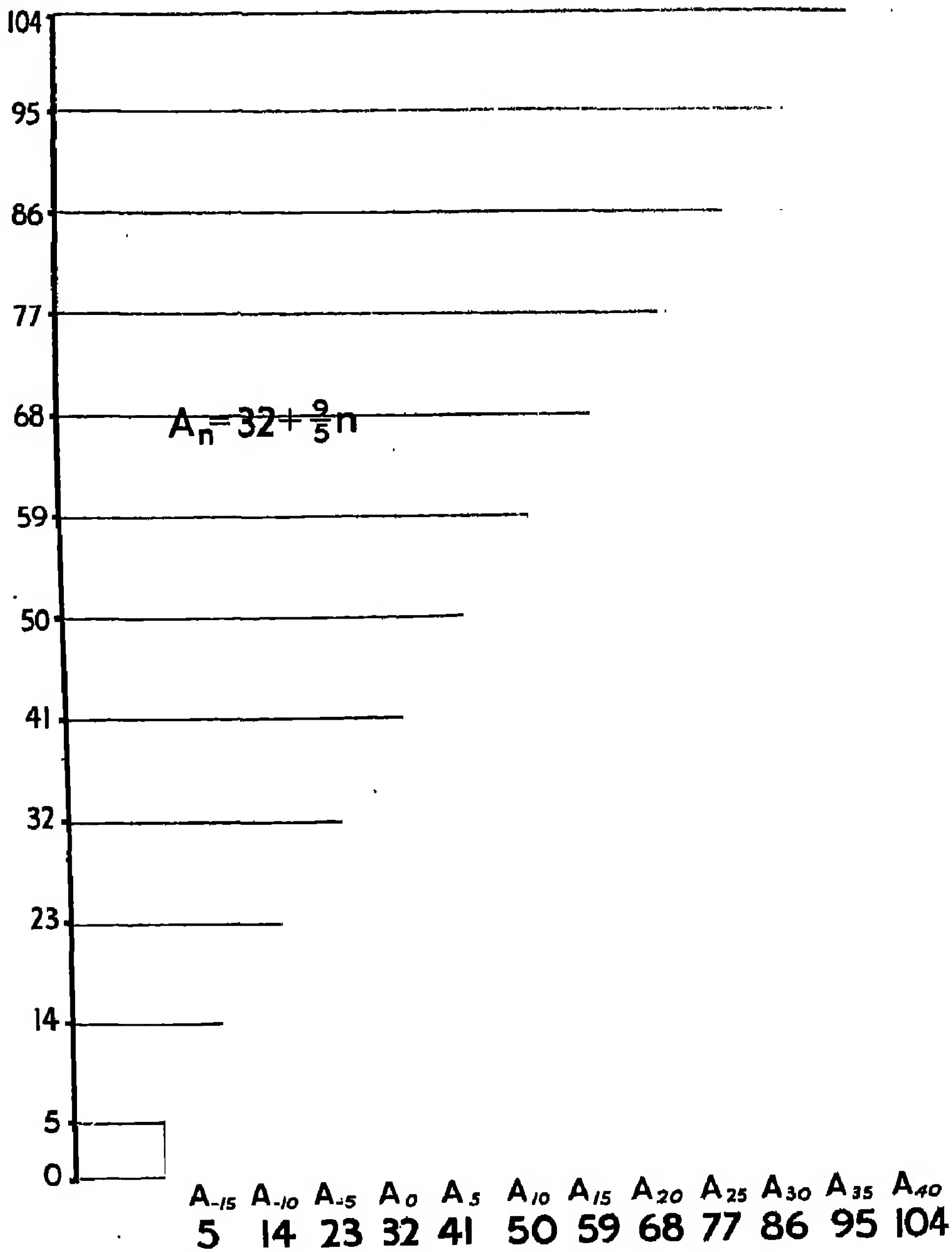
15. The first and nth terms of an arithmetic progression are a and l. Prove that the sum of the first n terms is $\frac{n}{2}(a+l)$.
Find the sum of all the positive integers less than 1,000, which are multiples of 7.

16. If a is the first term, l the last term, and S the sum of an arithmetic progression of n terms, prove that $S = \frac{n}{2}(a+l)$.
Find an expression for d, the common difference in terms of a, n and S.
The sum of 21 terms of an arithmetic progression whose first term is 20 is -840. Find the common difference. (L.G.S.)



$A_n = 8 + 3n$

Chart 44



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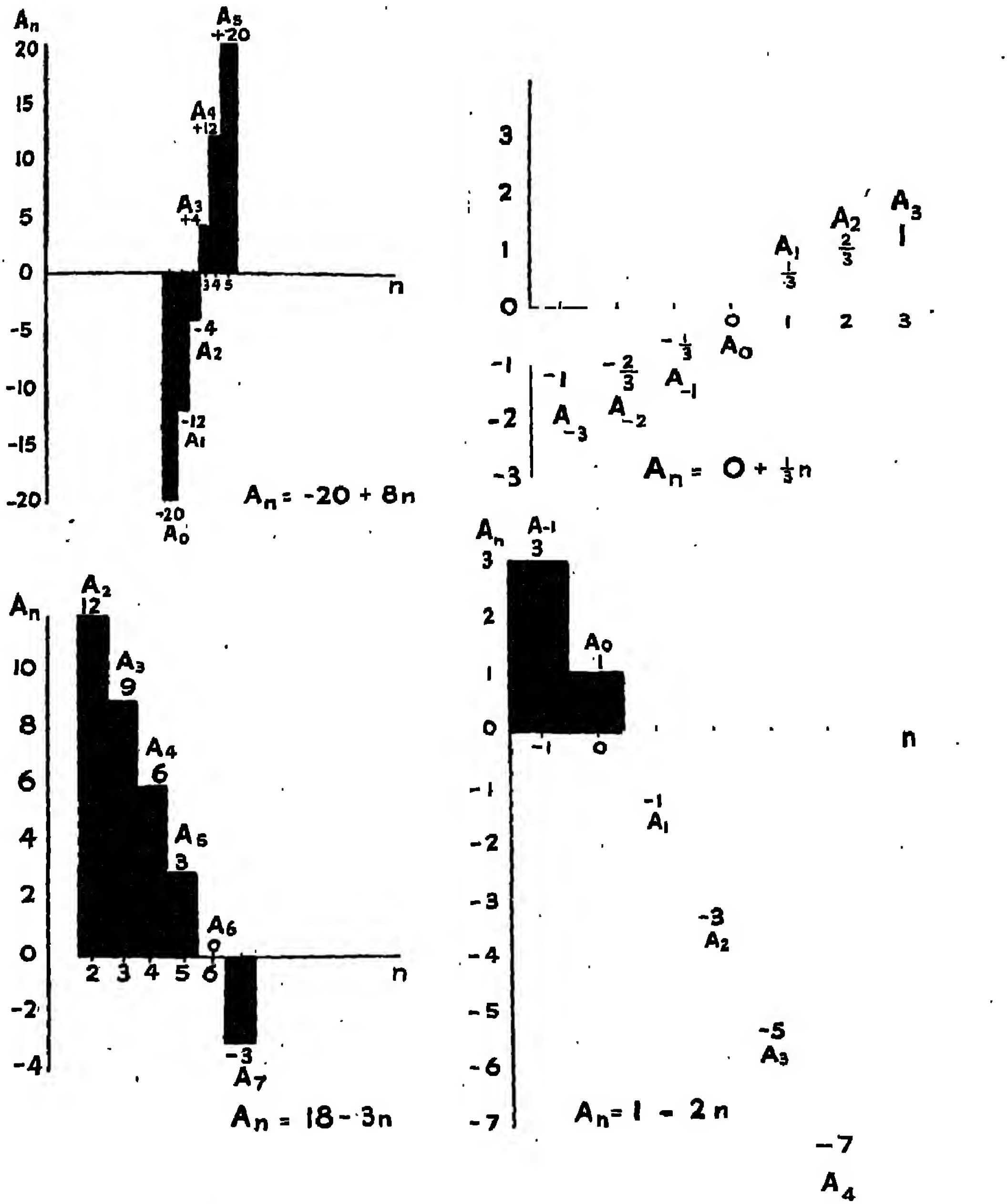
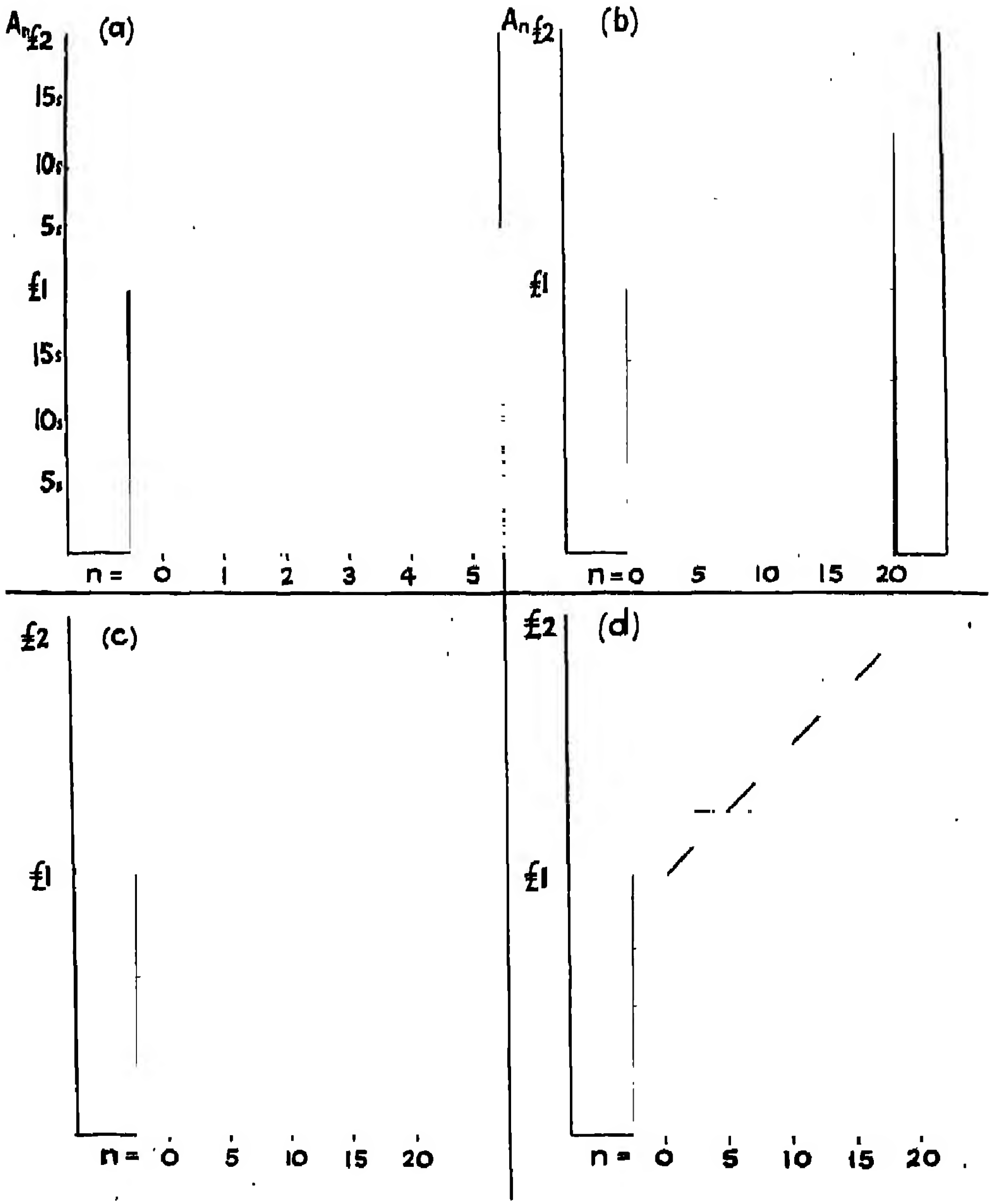
5
0
$A_n = 32 + \frac{9}{5}n$
A_{-15} A_{-10} A_{-5} A_0 A_5 A_{10} A_{15} A_{20} A_{25} A_{30} A_{35} A_{40}
5 14 23 32 41 50 59 68 77 86 95 104

Chart 45



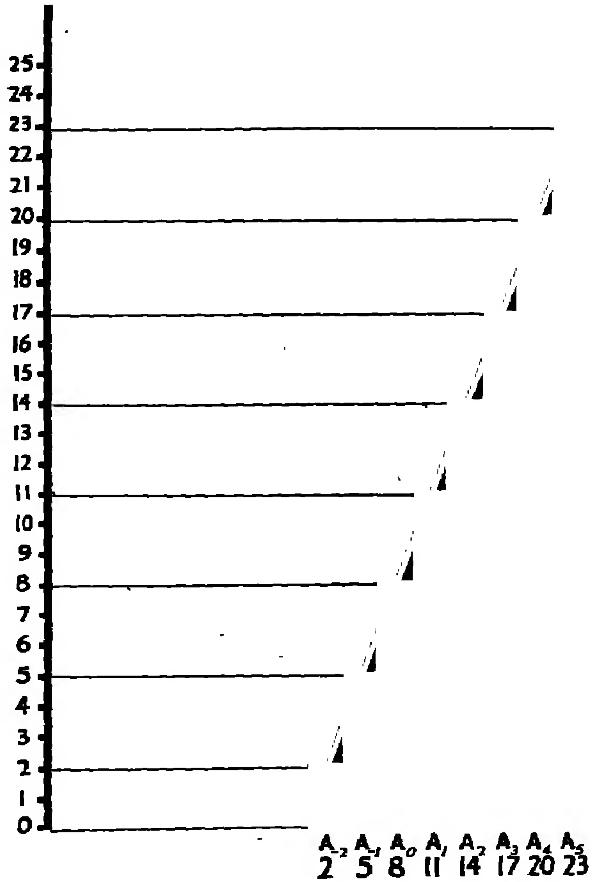
An
A5 +20
A4 +12
A3 +4
-4 A2
-12 A1
-20 A0
n
An = -20 + 8n
A3 1
A2 2/3
A1 1/3
A0
-1/3 A-1
-2/3 A-2
-1 A-3
An = 0 + 1/3 n
An
A2 12
A3 9
A4 6
A5 3
A6 0
-3 A7
n
An = 18 - 3n
An
A-1 3
A0 1
n
-1 A1
-3 A2
-5 A3
-7 A4
An = 1 - 2n

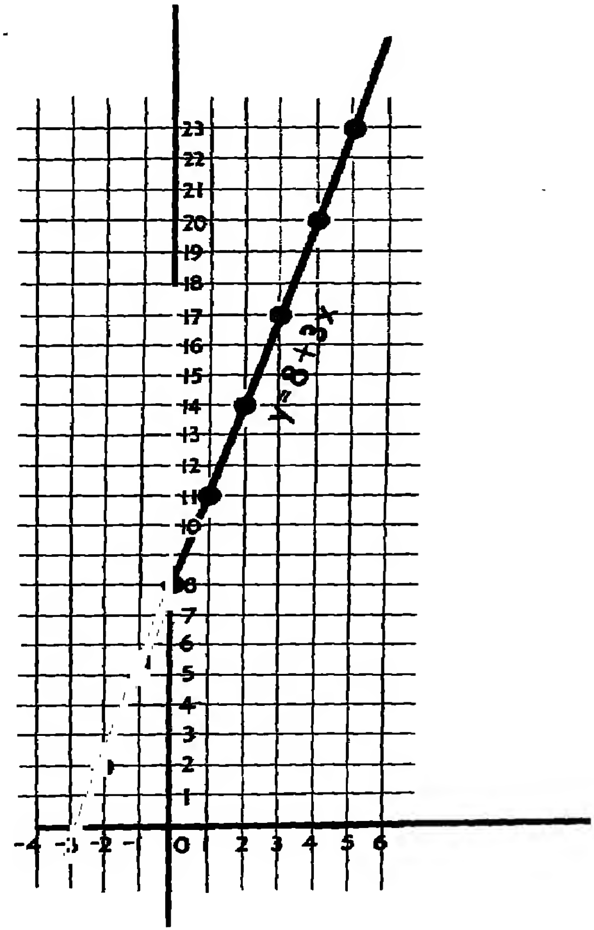
Chart 46



(a)
A_n £2
15s
10s
5s
£1
15s
10s
5s
n = 0 1 2 3 4 5
(b)
A_n £2
£1
n = 0 5 10 15 20
(c)
£2
£1
n = 0 5 10 15 20
(d)
£2
£1
n = 0 5 10 15 20

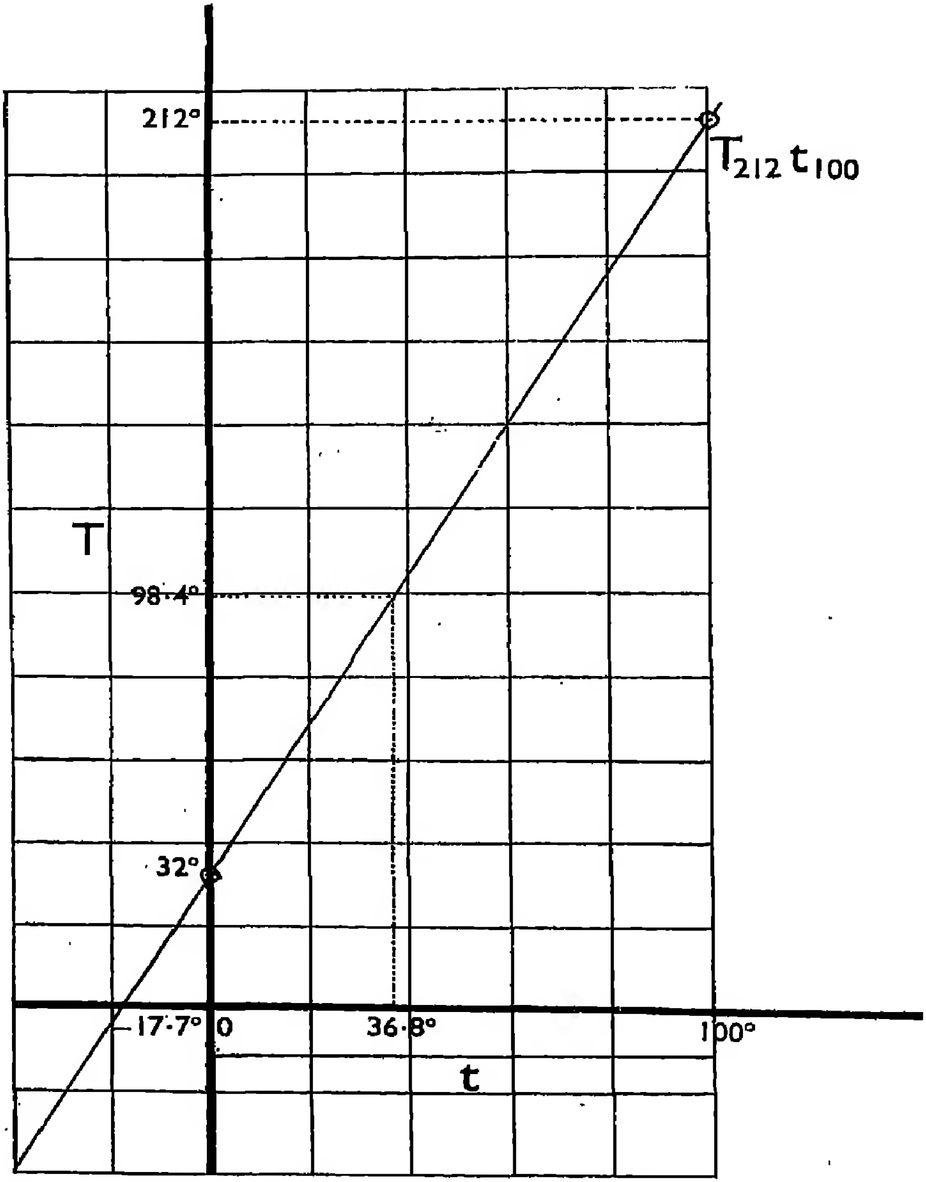
Chart 47





$\underline{A_n = 8+3n}$

Chart 48



$$T = \frac{9t}{5} + 32$$

Chart 49

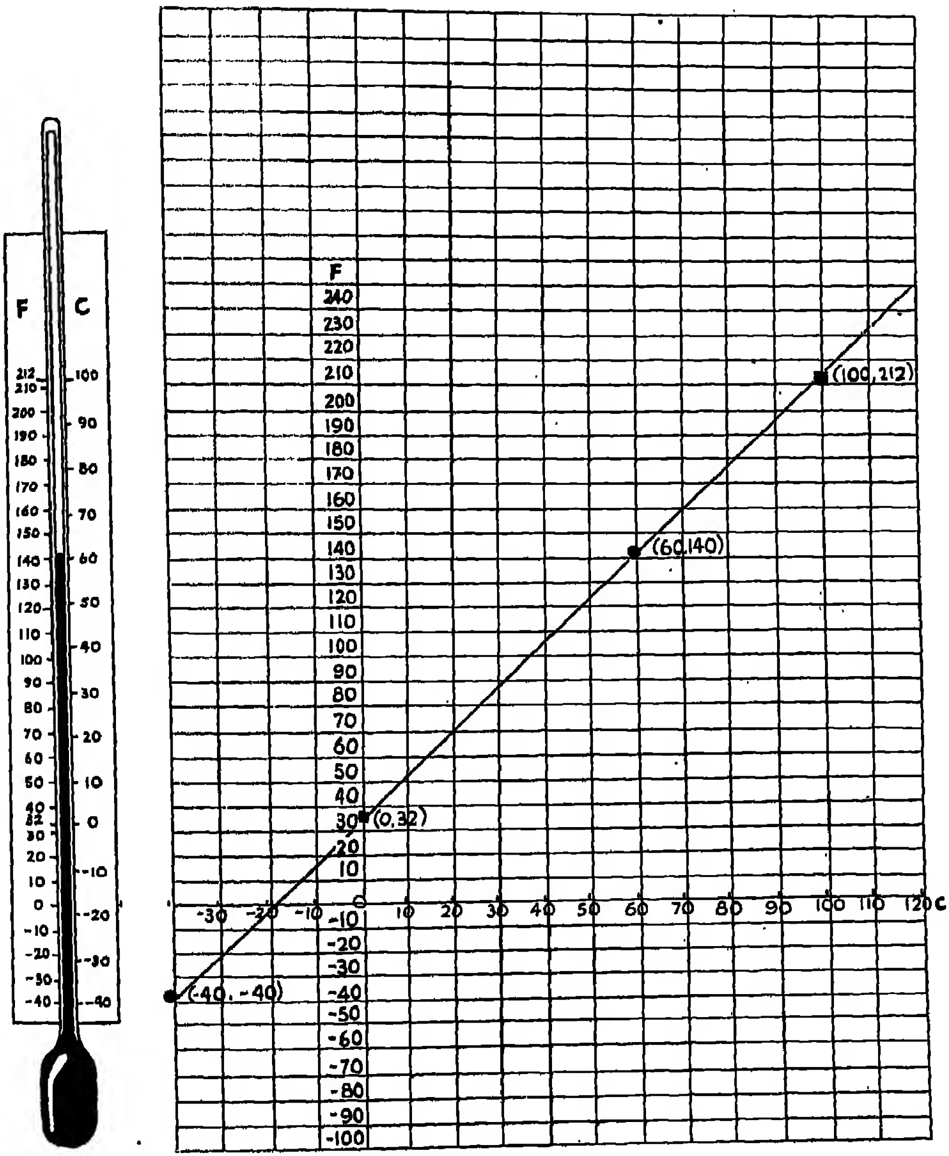


Chart 50

Book Two

b

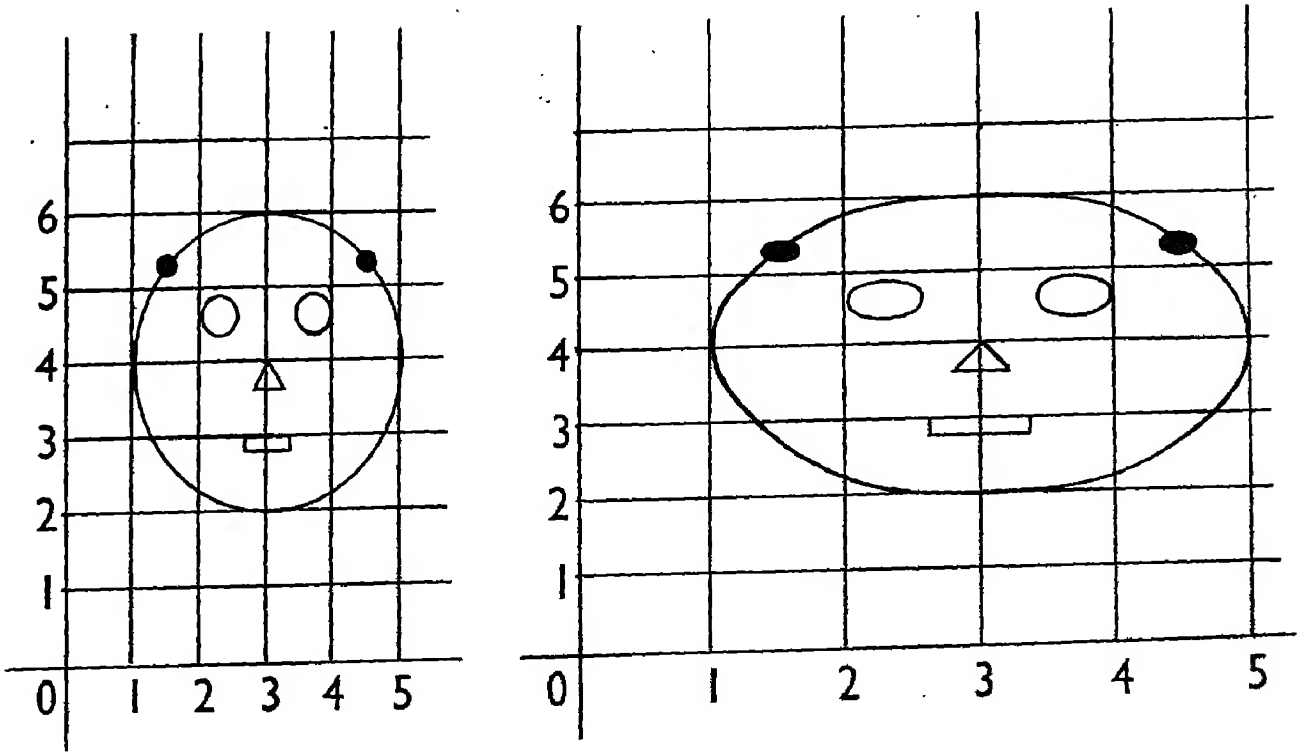


Chart 51

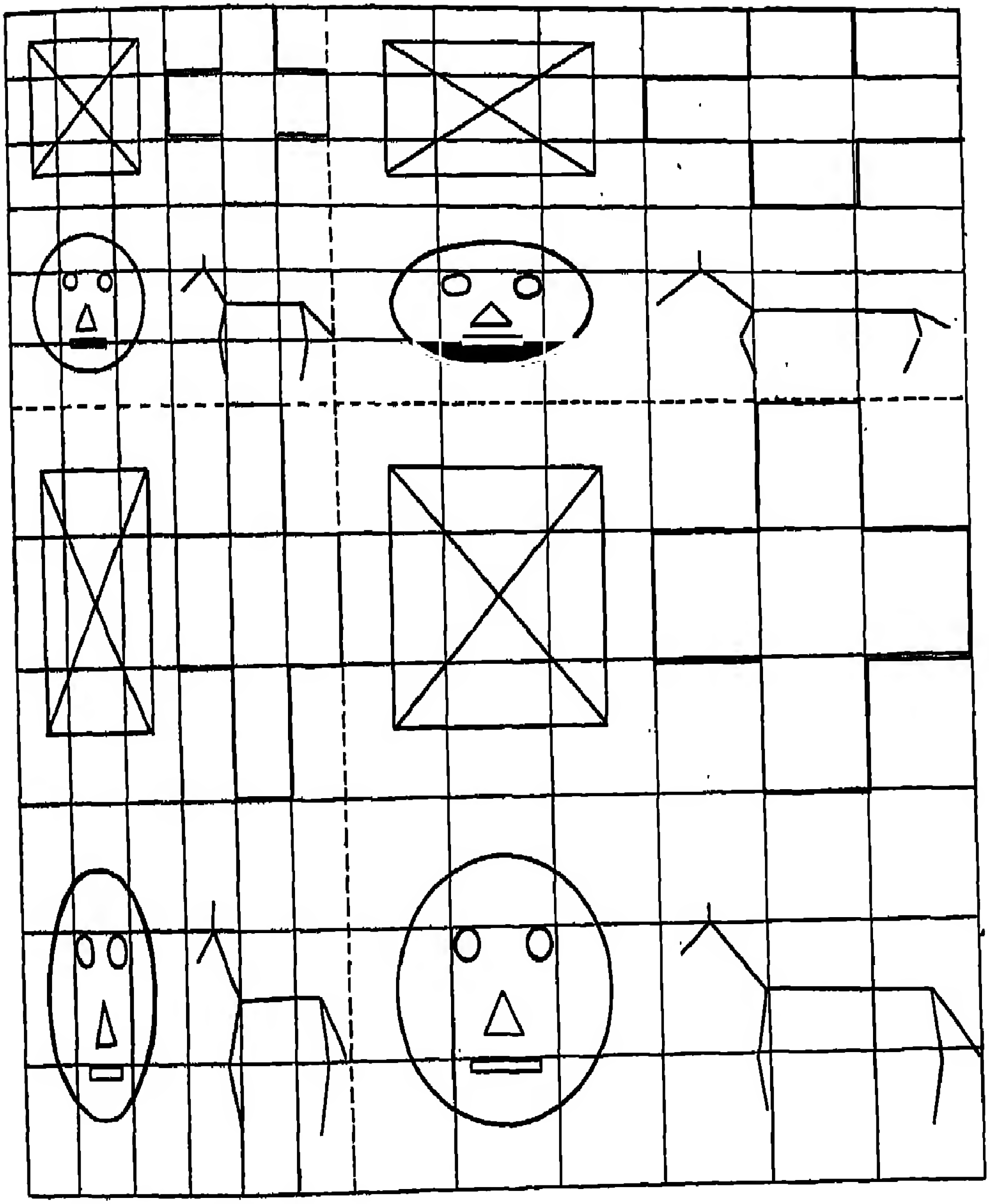


Chart 52

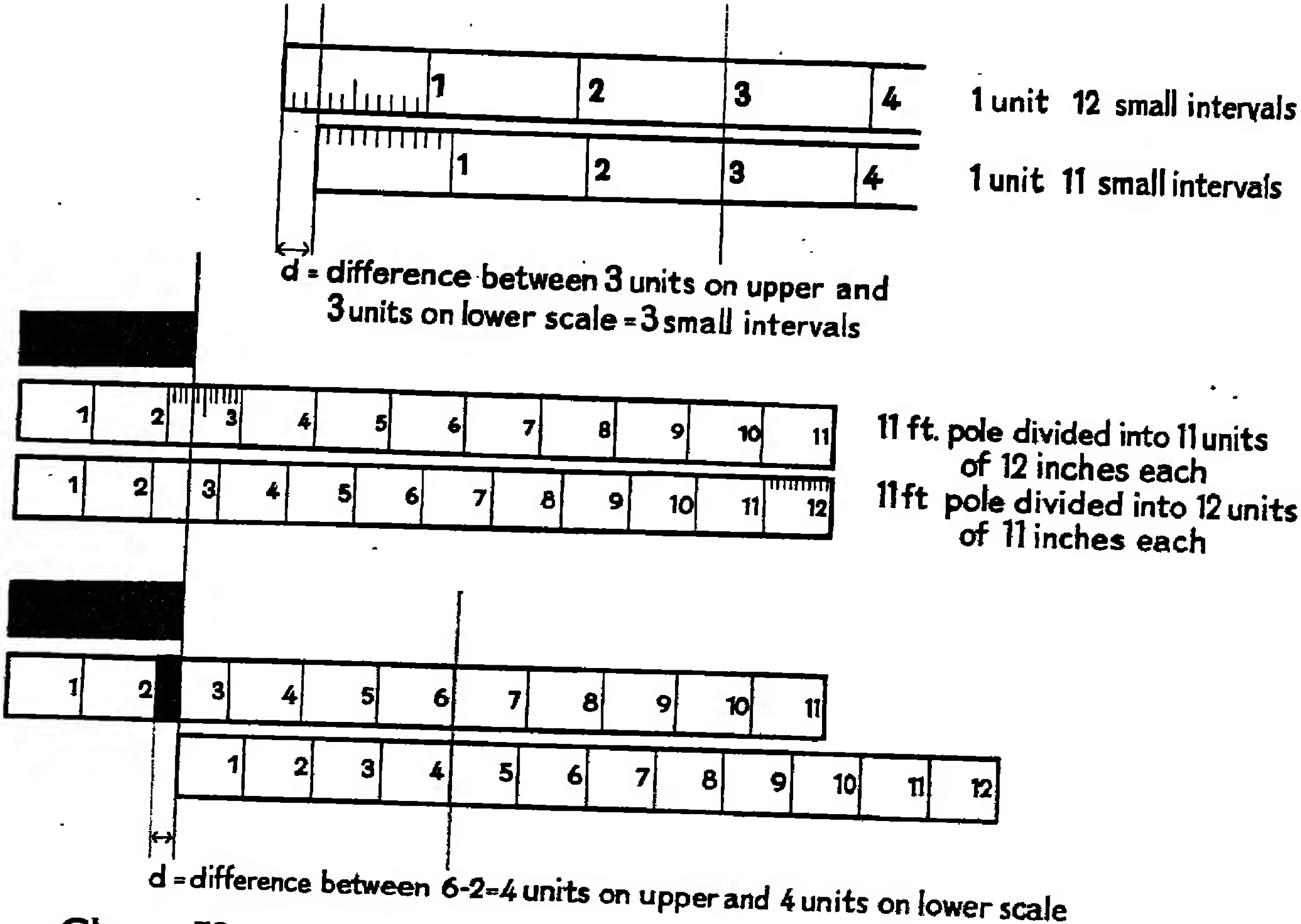
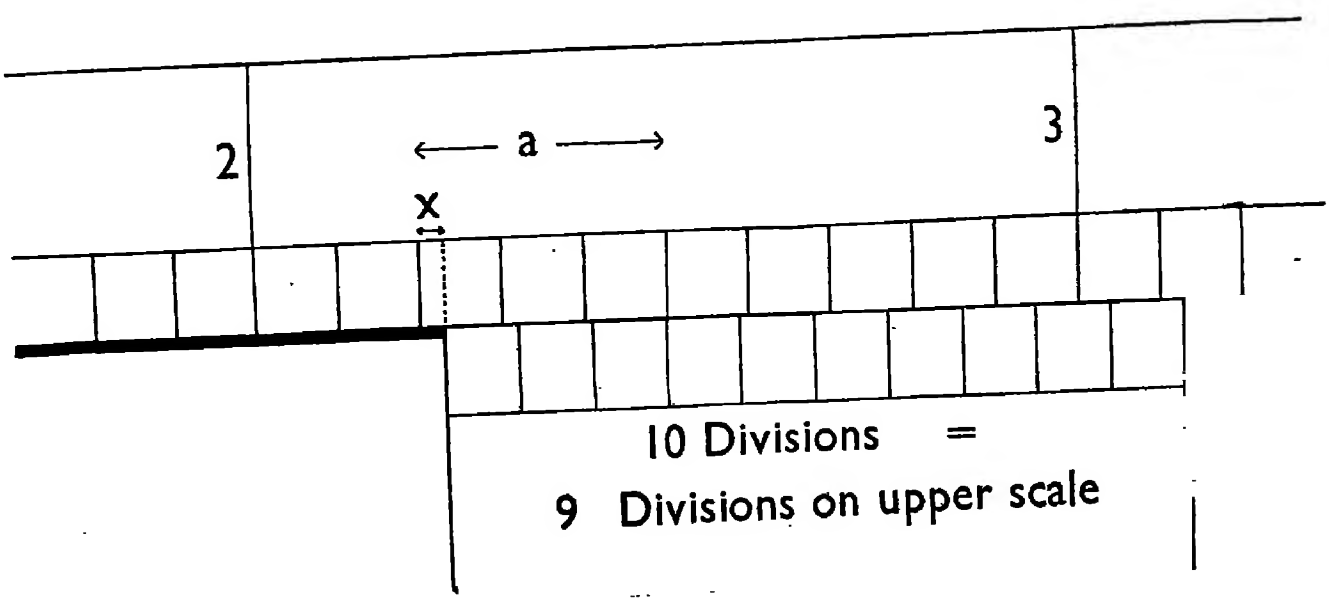

1 unit 12 small intervals
1 unit 11 small intervals
d = difference between 3 units on upper and 3 units on lower scale = 3 small intervals
11 ft. pole divided into 11 units of 12 inches each
11 ft pole divided into 12 units of 11 inches each
d = difference between 6-2=4 units on upper and 4 units on lower scale

Chart 53

The Vernier Scale

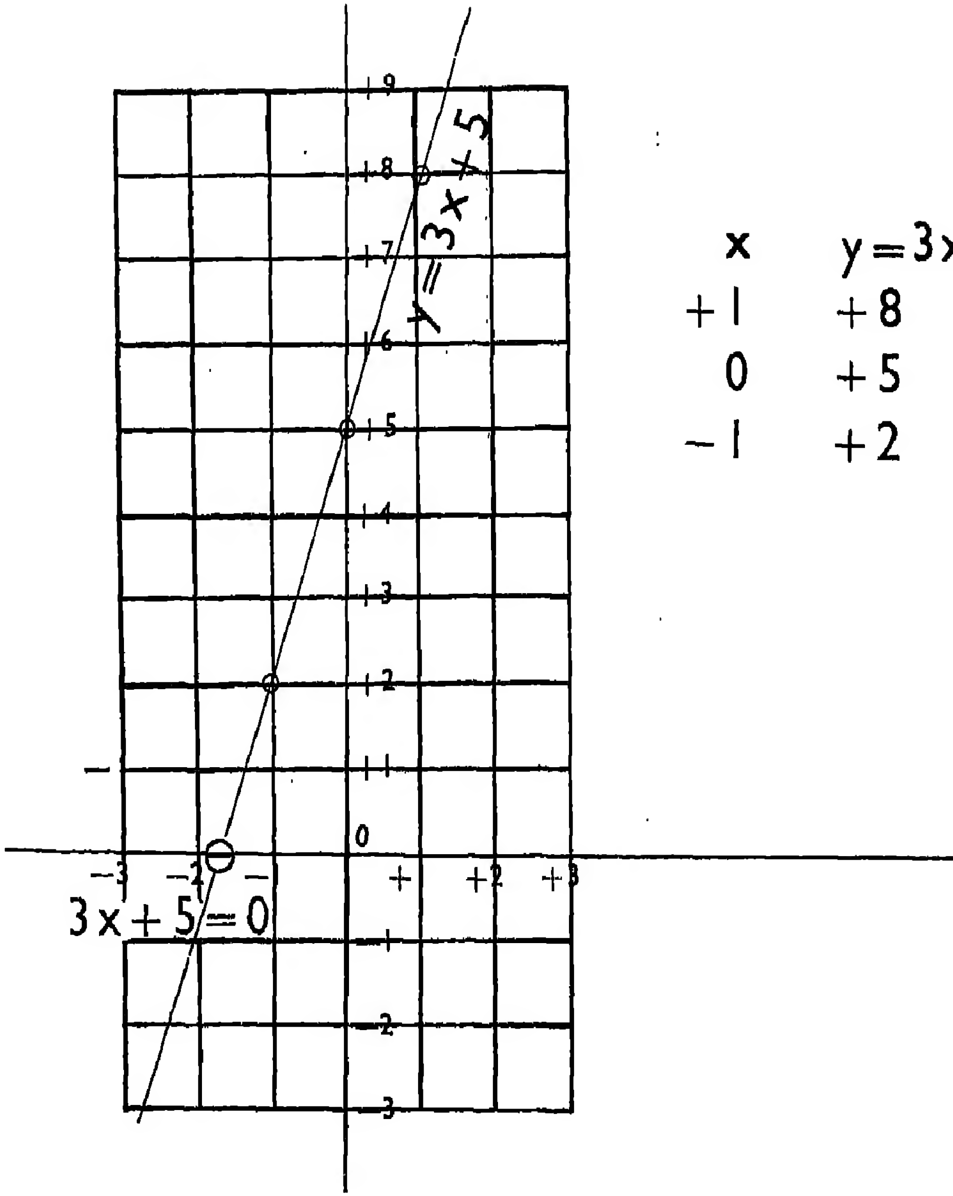


10 Divisions =
9 Divisions on upper scale

$$\left(\tfrac{9}{10}a\right) + x = a$$

$$\therefore x = \tfrac{1}{10}a$$

Chart 54



x	$y=3x+5$
+1	+8
0	+5
−1	+2

Chart 55

LINEAR FUNCTIONS of SPECIAL INTEREST:

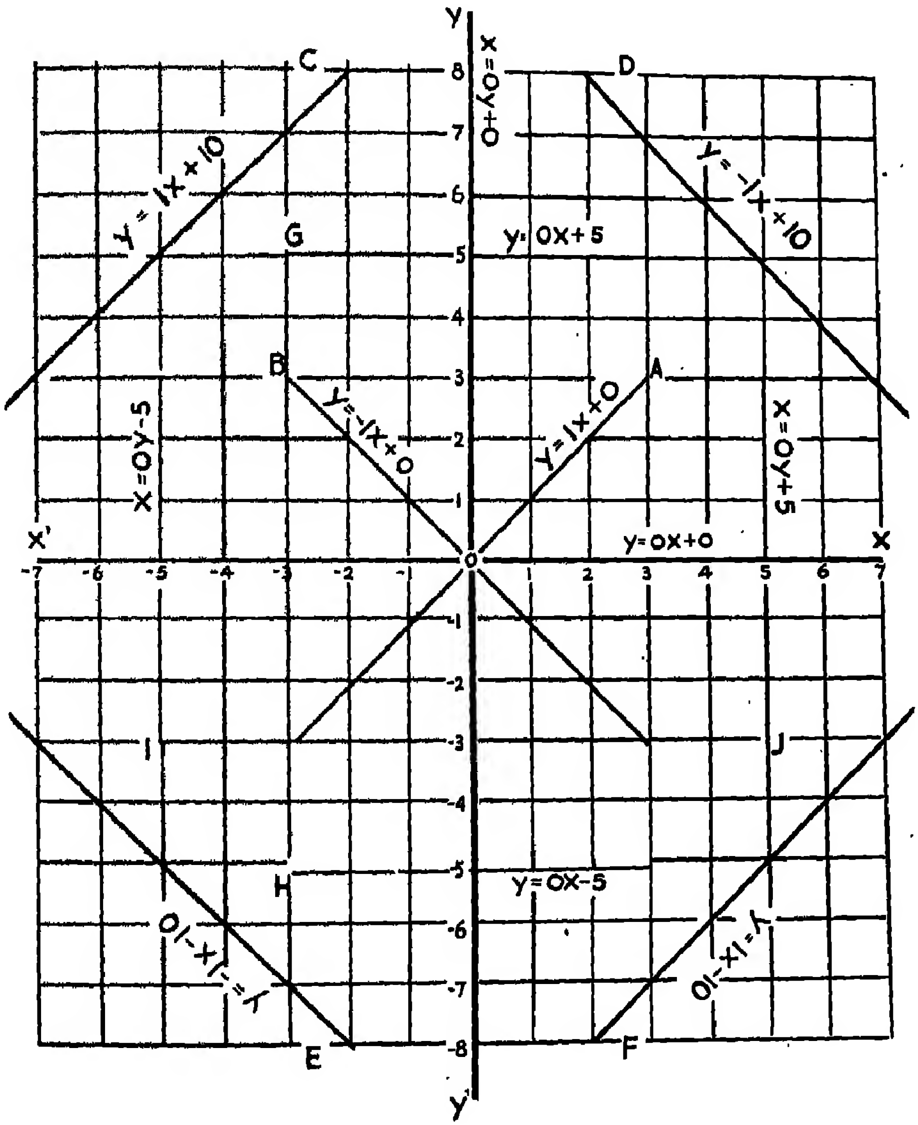


Chart 56

EPISODES in the CAREER of the FUNCTION $y=2x+3$

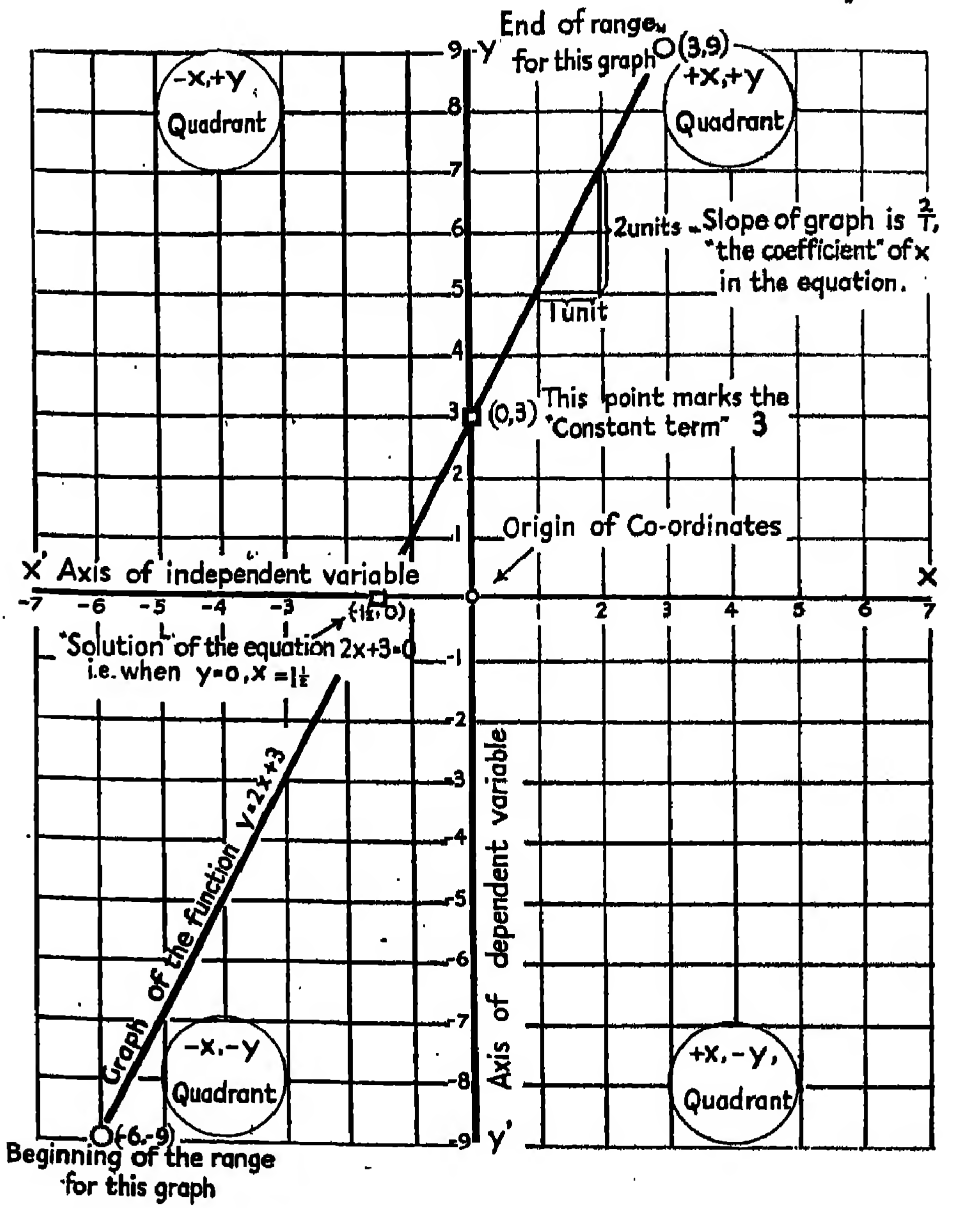


Chart 57

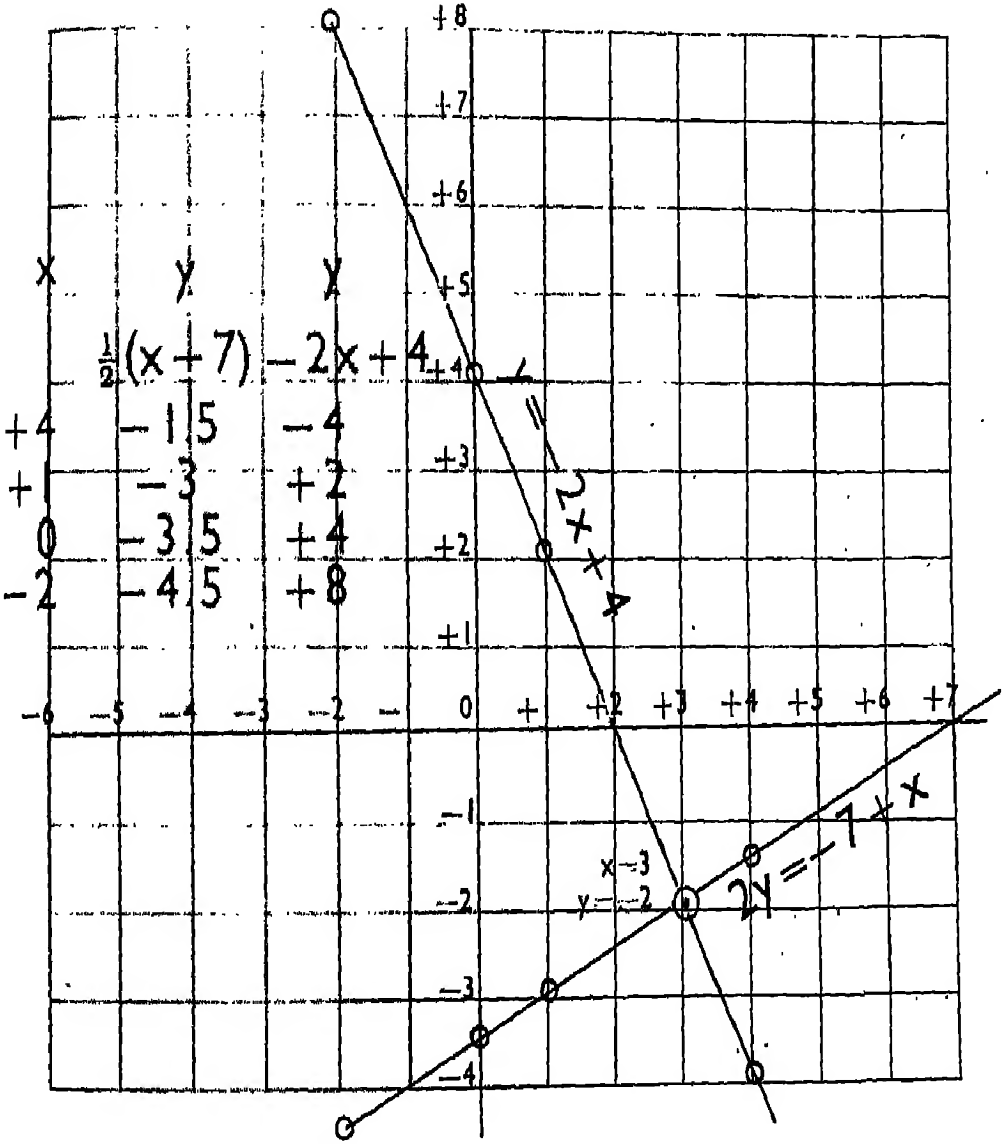
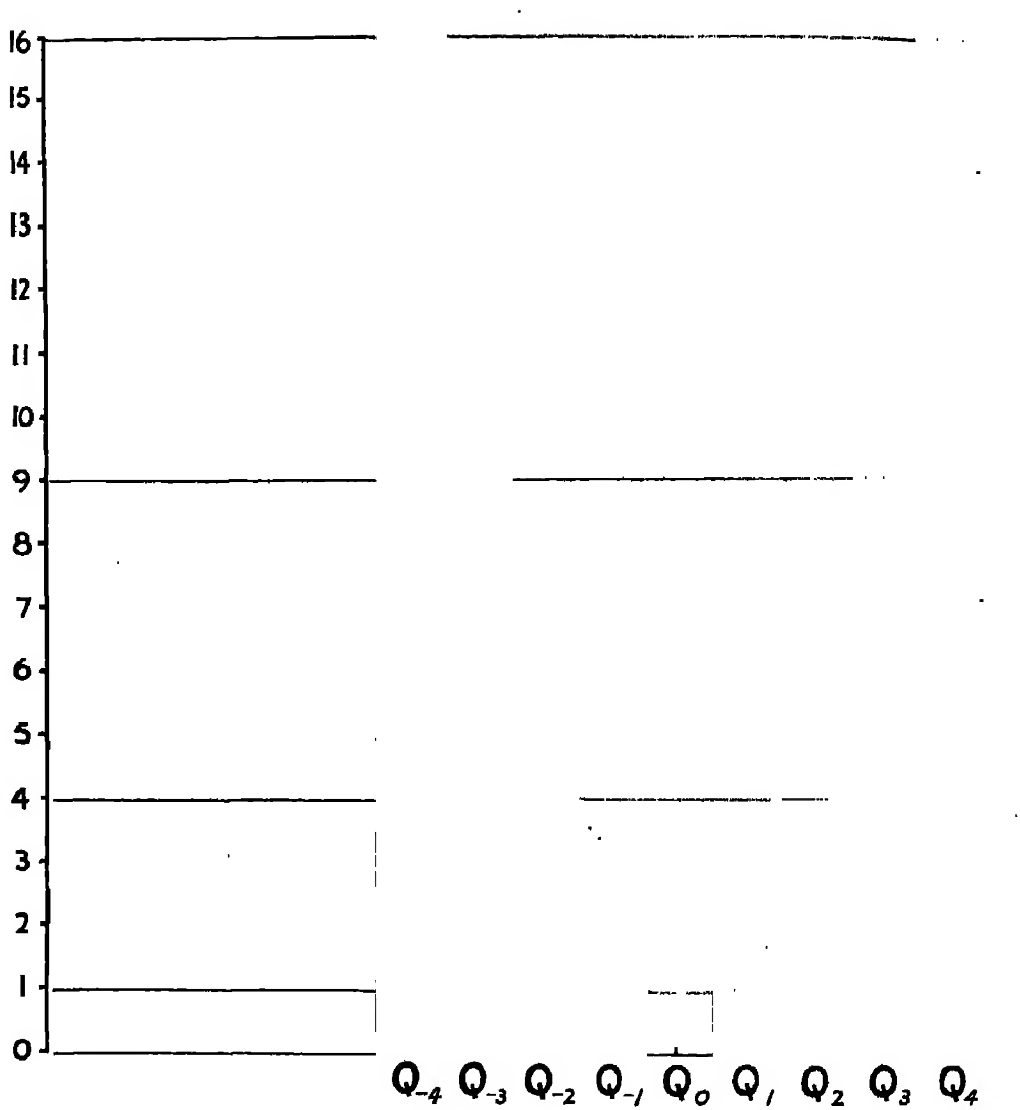

x
y
y
$\frac{1}{2}(x+7)$
$-2x+4$
+4 −1.5 −4
+1 −3 +2
0 −3.5 +4
−2 −4.5 +8
$y=-2x+4$
$2y=-7+x$
x=3
y=−2
+8
+7
+6
+5
+4
+3
+2
+1
0
−1
−2
−3
−4
−6
−5
−4
−3
−2
+2
+3
+4
+5
+6
+7

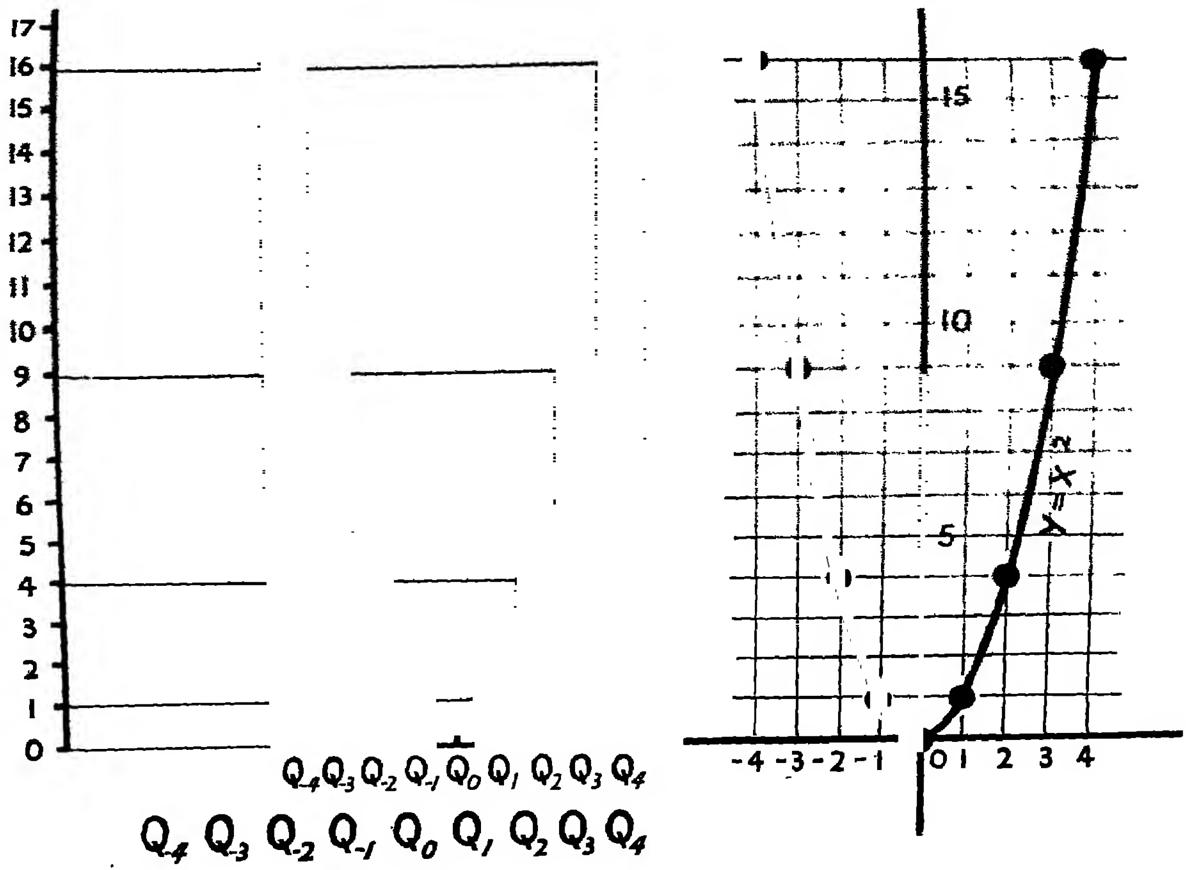
Chart 58



Q_{-4}	Q_{-3}	Q_{-2}	Q_{-1}	Q_0	Q_1	Q_2	Q_3	Q_4
16	9	4	1	0	1	4	9	16

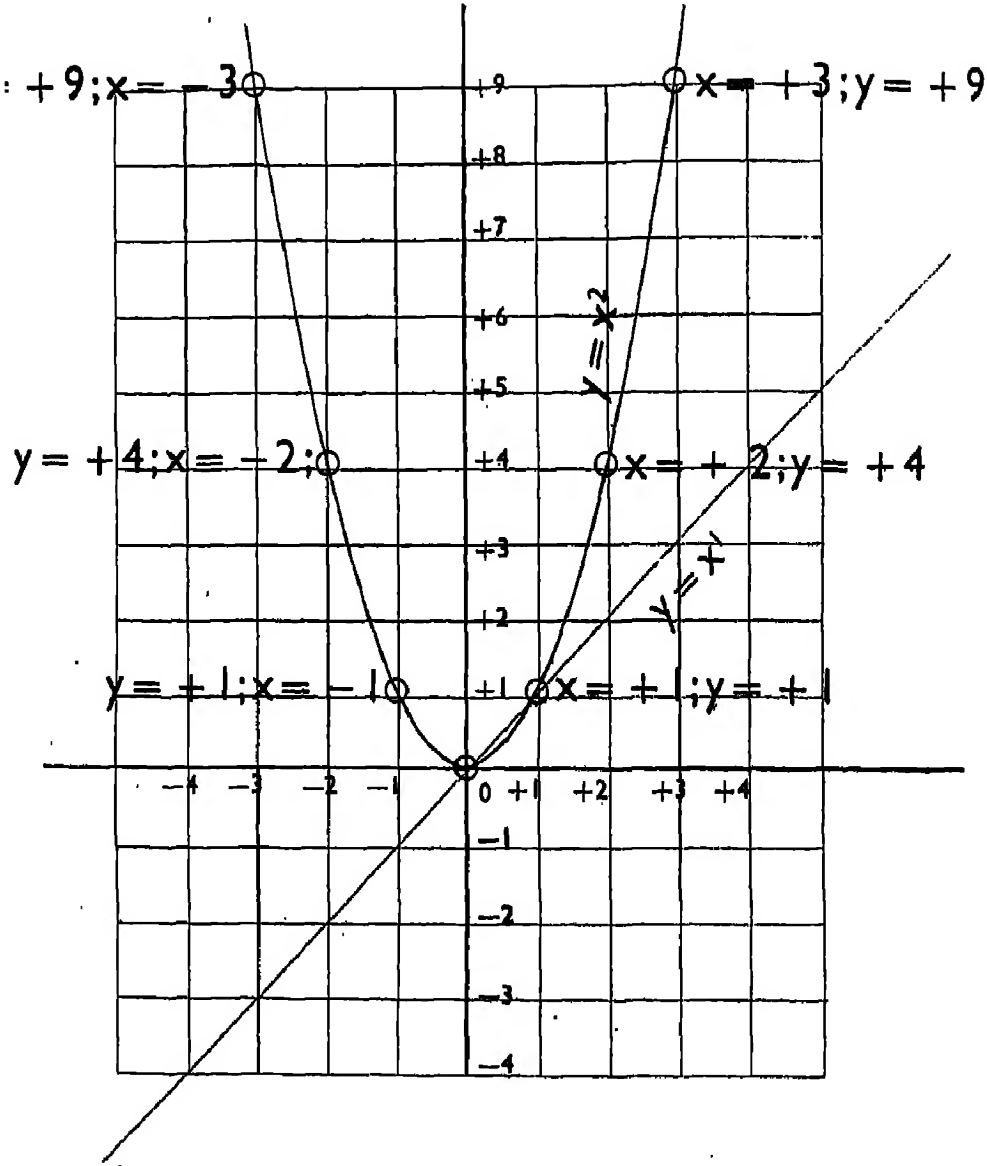
$$Q_n = n^2$$

Chart 59



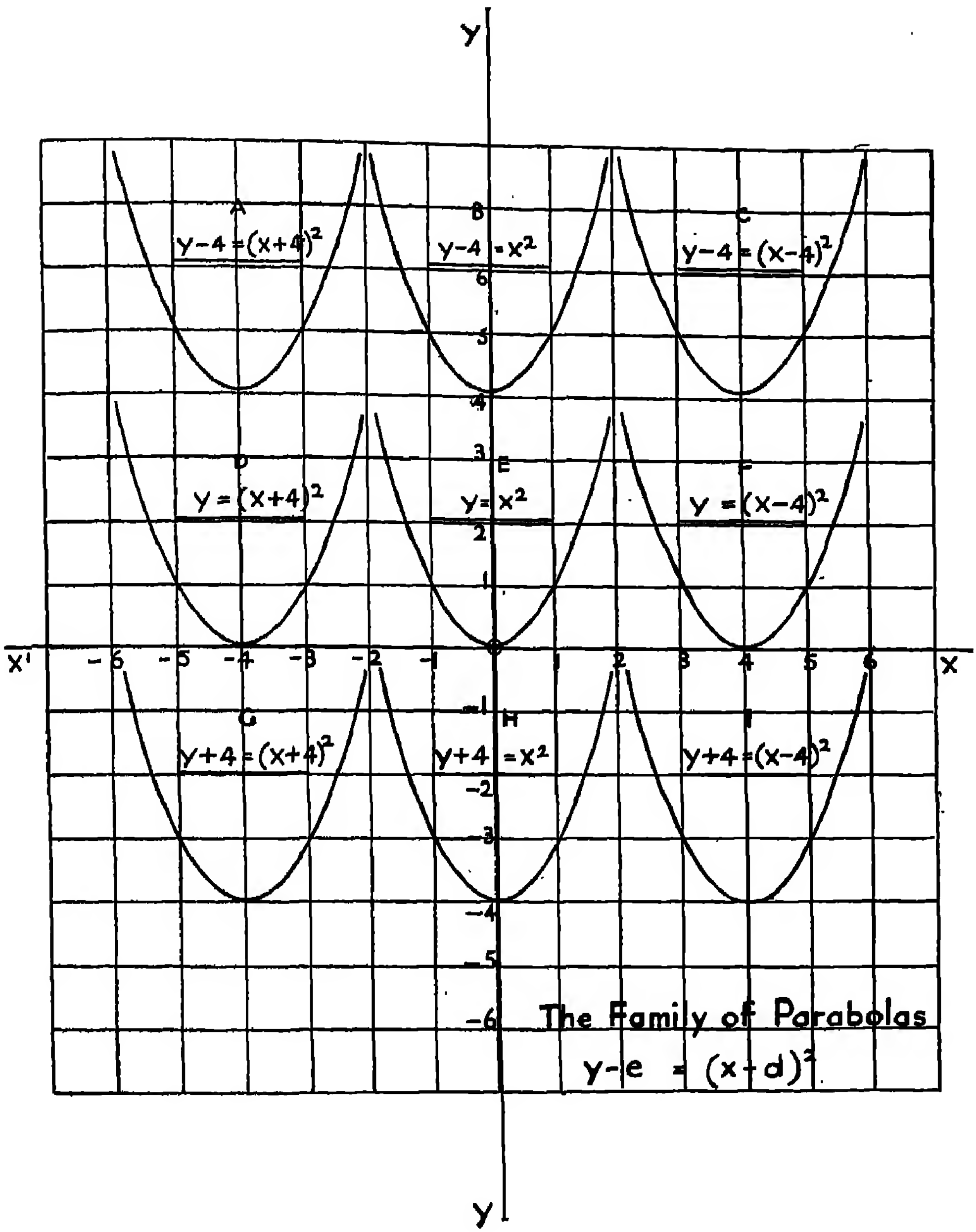
$Q_n = n^2$

Chart 60



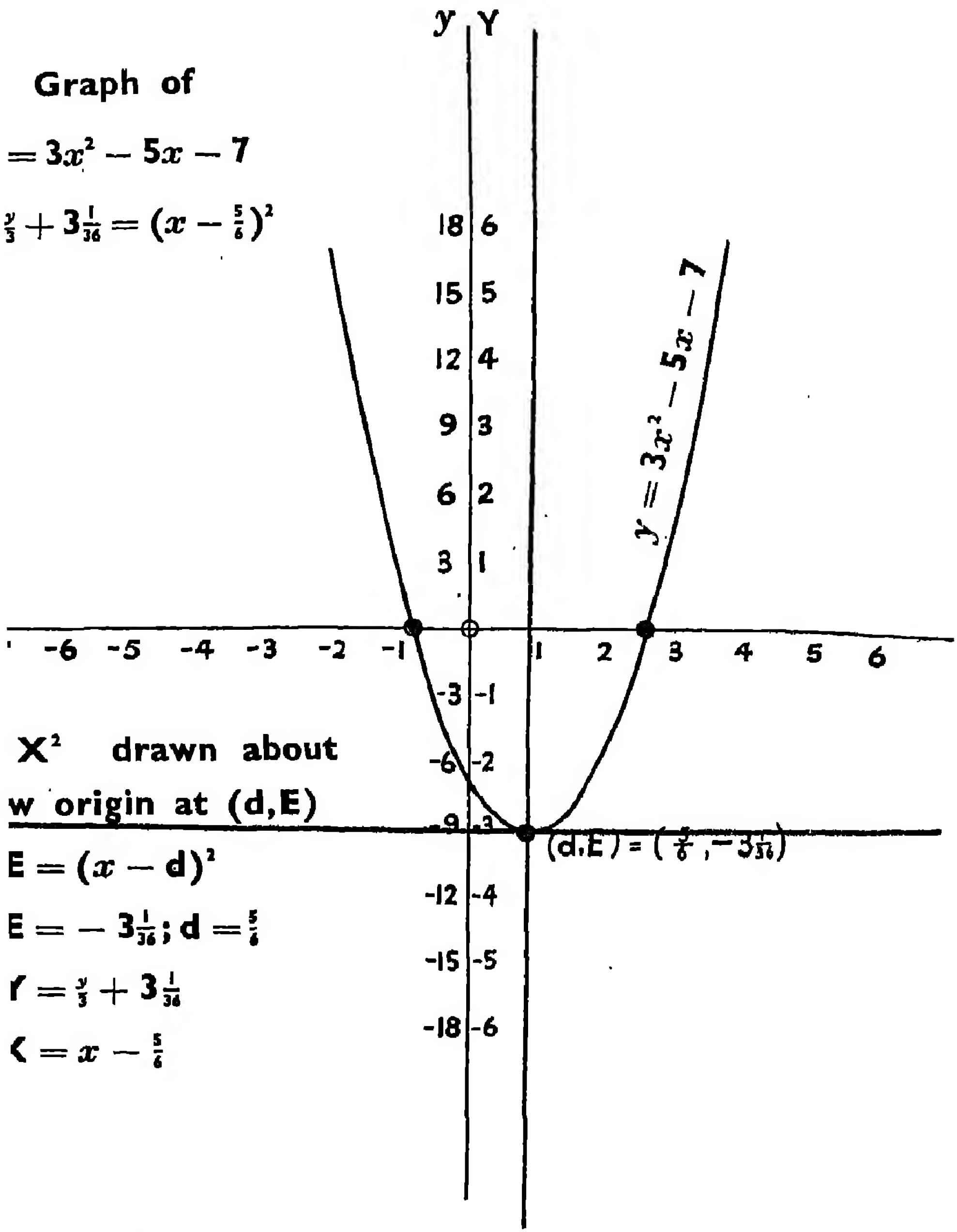
+9;x=−3
x=+3;y=+9
y=+4;x=−2;
x=+2;y=+4
y=+1;x=−1
x=+1;y=+1
$y=x^2$
$y=x$
+9
+8
+7
+6
+5
+4
+3
+2
+1
−4 −3 −2 −1 0 +1 +2 +3 +4
−1
−2
−3
−4

Chart 61



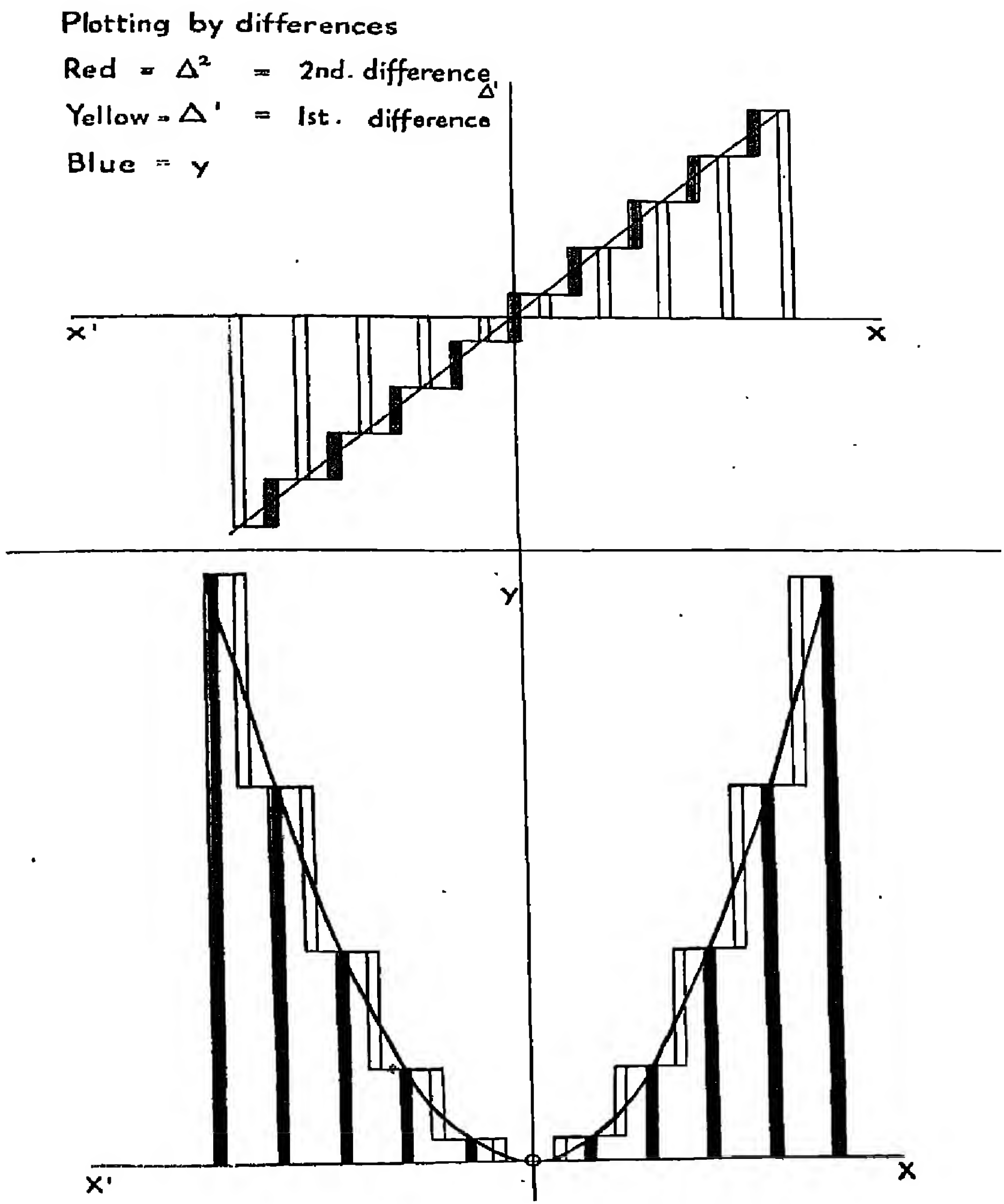
The Family of Parabolas
y-e = (x+d)²
A
y-4 = (x+4)²
B
y-4 = x²
C
y-4 = (x-4)²
D
y = (x+4)²
E
y = x²
F
y = (x-4)²
G
y+4 = (x+4)²
H
y+4 = x²
I
y+4 = (x-4)²

Chart 62



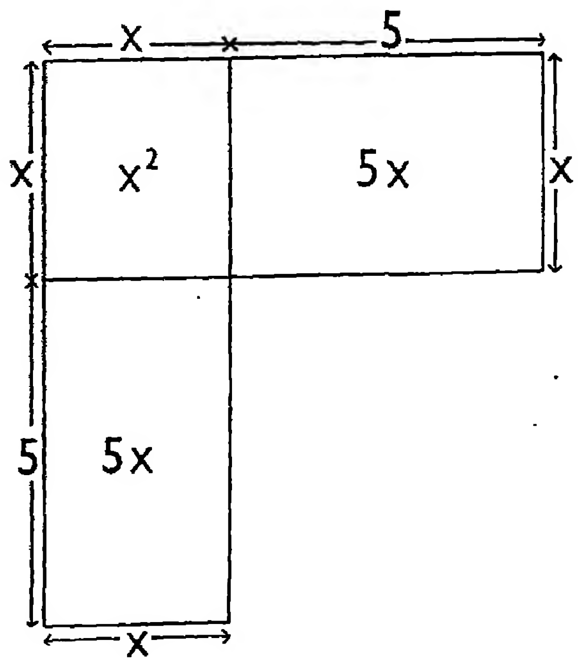
Graph of
$= 3x^2 - 5x - 7$
$\frac{y}{3} + 3\frac{1}{36} = (x - \frac{5}{6})^2$
y Y
18 6
15 5
12 4
9 3
6 2
3 1
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
-3 -1
-6 -2
-9 -3
-12 -4
-15 -5
-18 -6
$y = 3x^2 - 5x - 7$
$(d,E) = (\frac{5}{6}, -3\frac{1}{36})$
X^2 drawn about
w origin at (d,E)
$E = (x - d)^2$
$E = -3\frac{1}{36}$; $d = \frac{5}{6}$
$\frac{y}{3} + 3\frac{1}{36}$
$= x - \frac{5}{6}$

hart 63

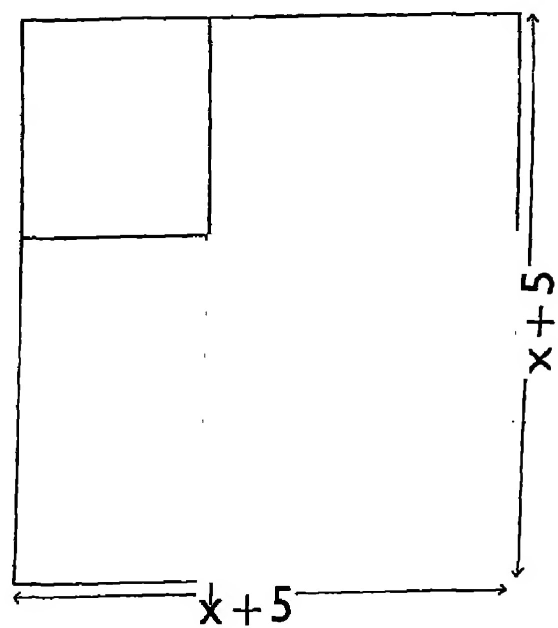


Plotting by differences
Red = Δ² = 2nd. difference
Yellow = Δ' = Ist. difference
Blue = y
Δ'
X'
X
Y
X'
X

Chart 64



$$x^2 + 10x = 39$$



$$x^2 + 10x + 25 = 39 + 25$$
$$(x+5)^2 = 64 = 8^2$$
$$x + 5 = 8$$

Chart 65

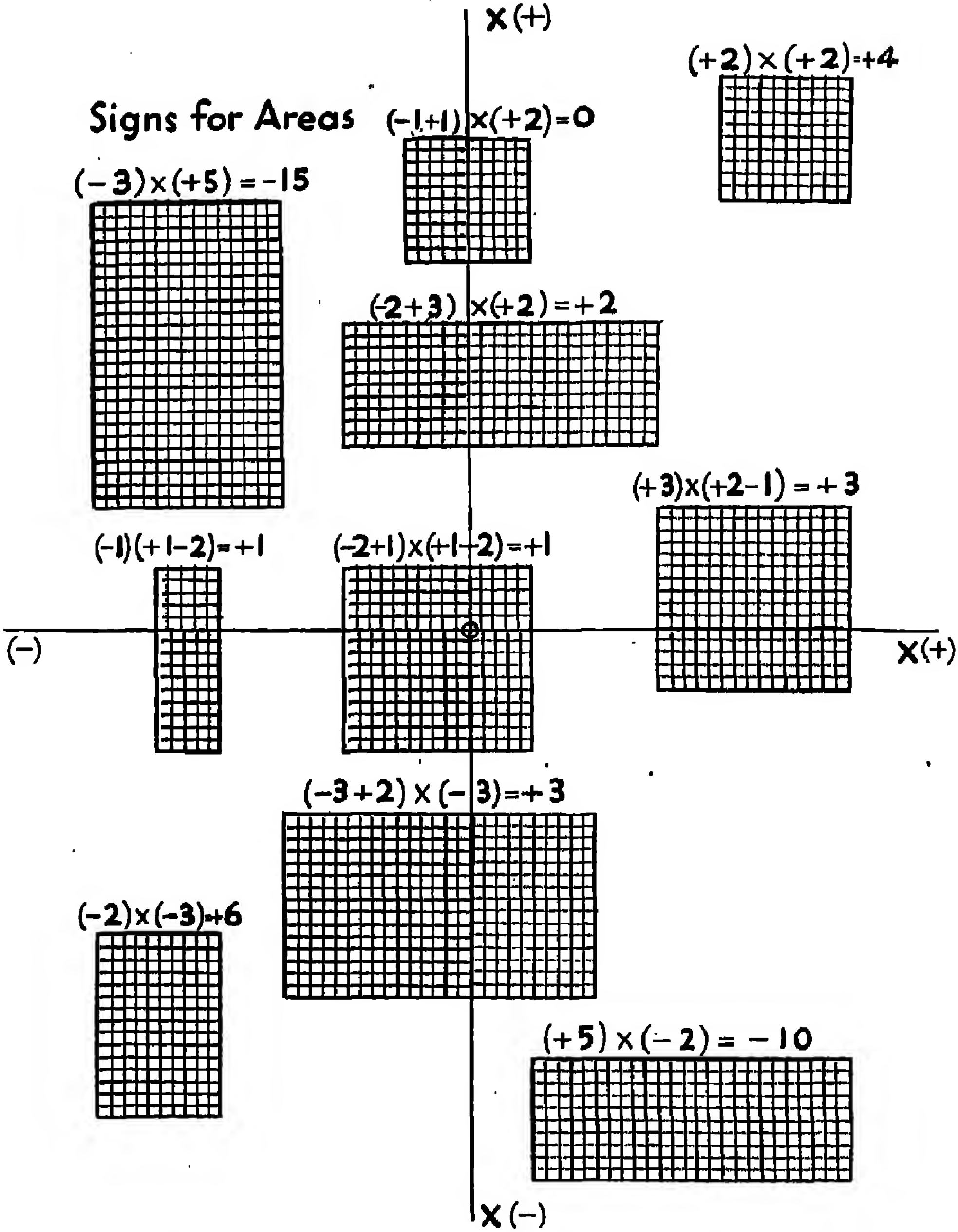
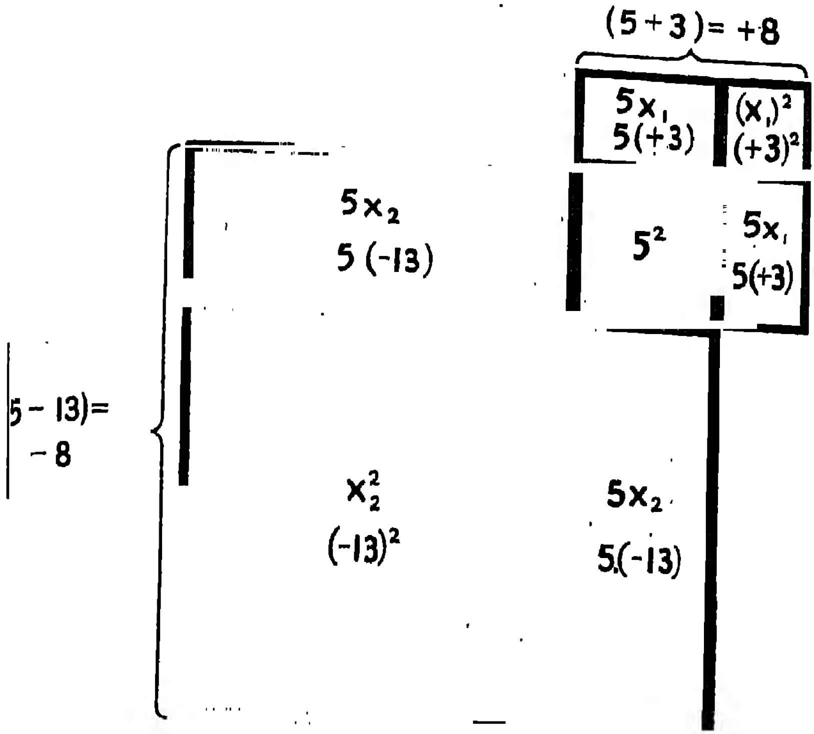

x (+)
(+2) x (+2) = +4
Signs for Areas
(−1+1) x (+2) = 0
(−3) x (+5) = −15
(−2+3) x (+2) = +2
(+3) x (+2−1) = +3
(−1)(+1−2) = +1
(−2+1) x (+1−2) = +1
(−)
x (+)
(−3+2) x (−3) = +3
(−2) x (−3) = +6
(+5) x (−2) = −10
x (−)

Chart 66

Geometric Representation of Quadratic

$$x^2 + 10x = 39$$
$$x^2 + 10x + 25 = 64$$
$$(x_1 + 5)^2 = (+8)^2 \quad (x_2 + 5) = (-8)^2$$
$$(3 + 5) = +8 \quad (-13 + 5) = -8$$



$$(+3)^2 + 10(+$$
$$= 9 + 30$$
$$= 39$$

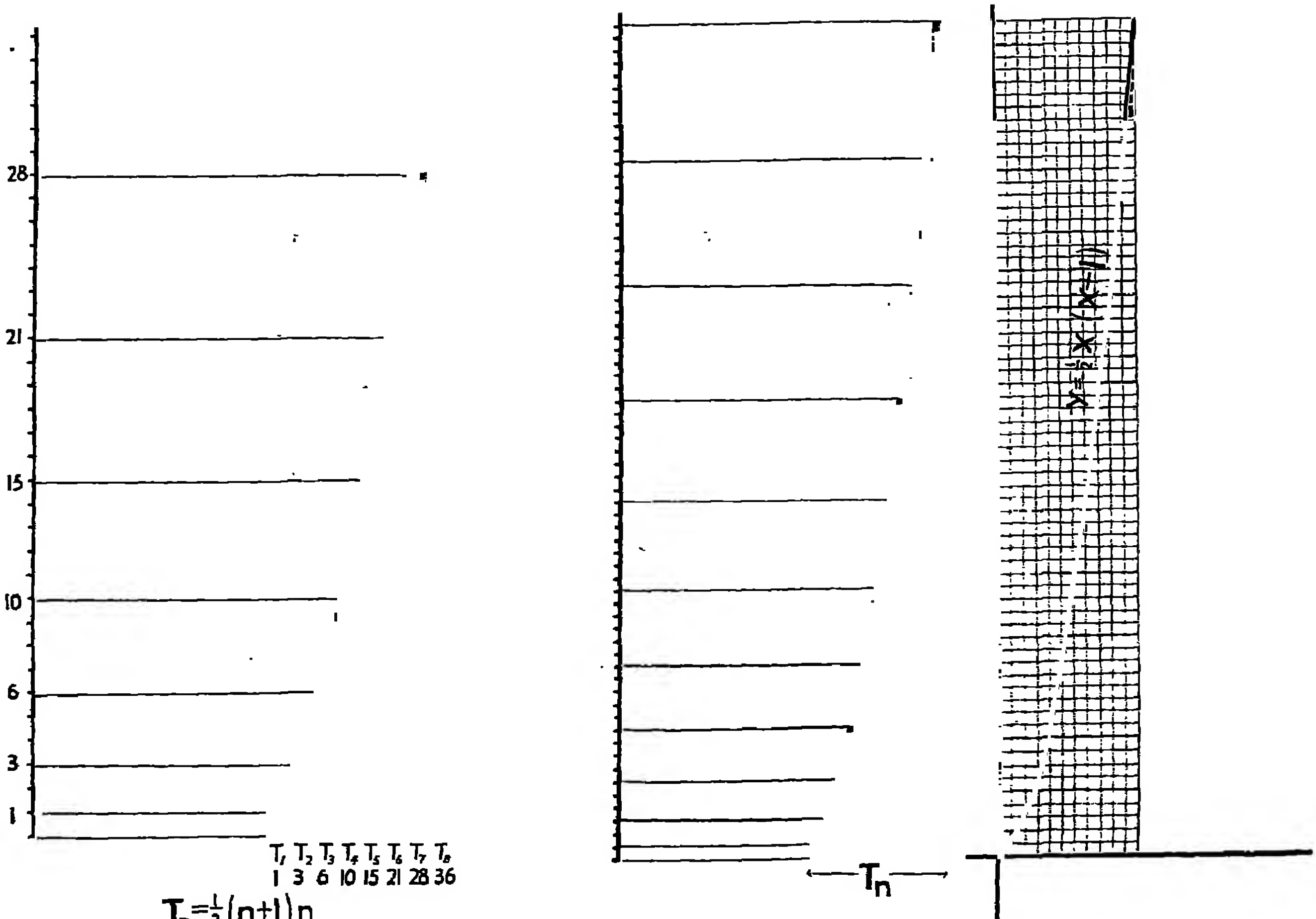
$$(-13)^2 + 10(-13)$$
$$= 169 - 130$$
$$= 39$$

Lines red +
blue −

Areas red-red +
blue-blue +
red-blue −
blue-red −

Chart 67

look Two



$T_n=\frac{1}{2}(n+1)n$

Chart 68

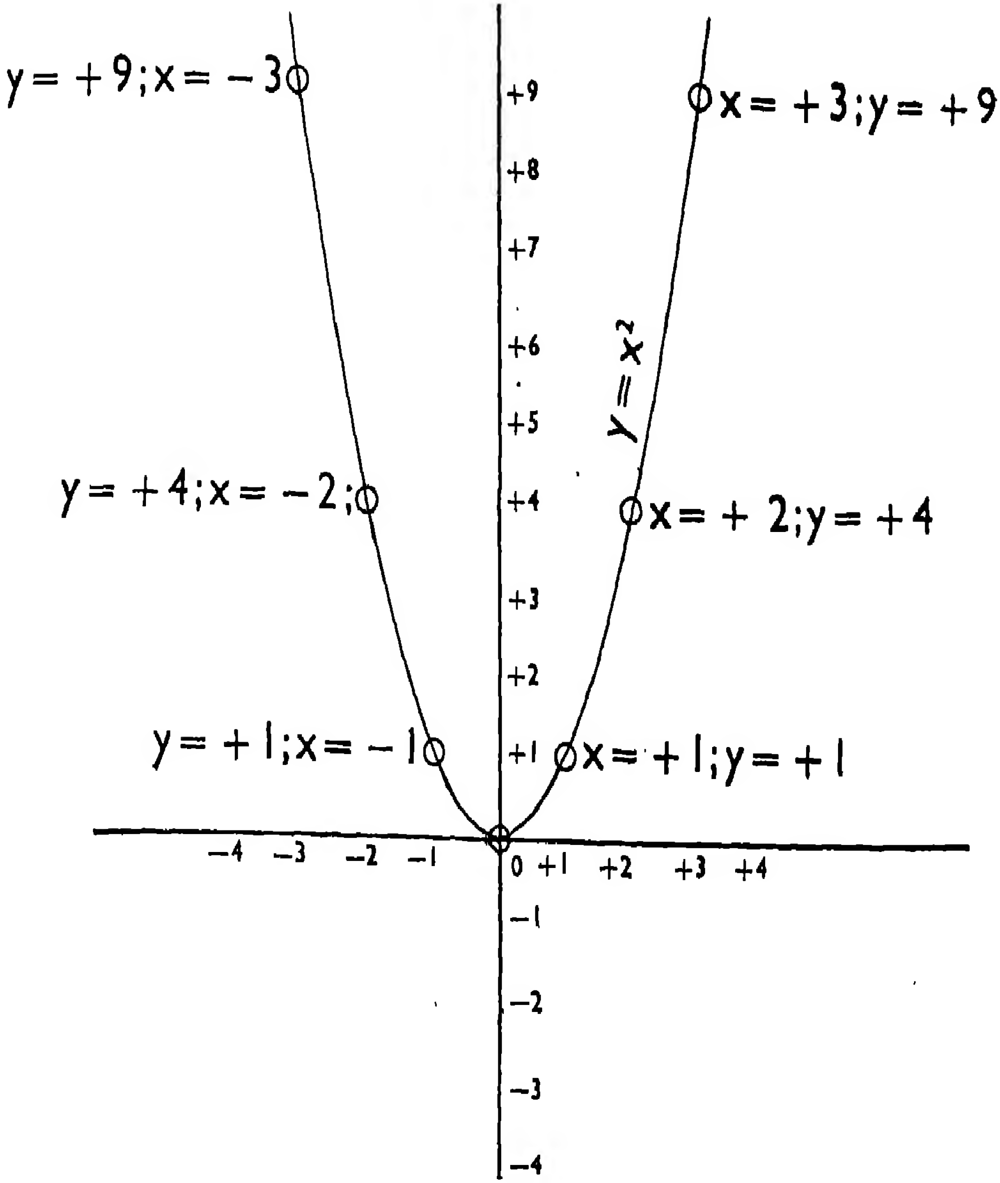
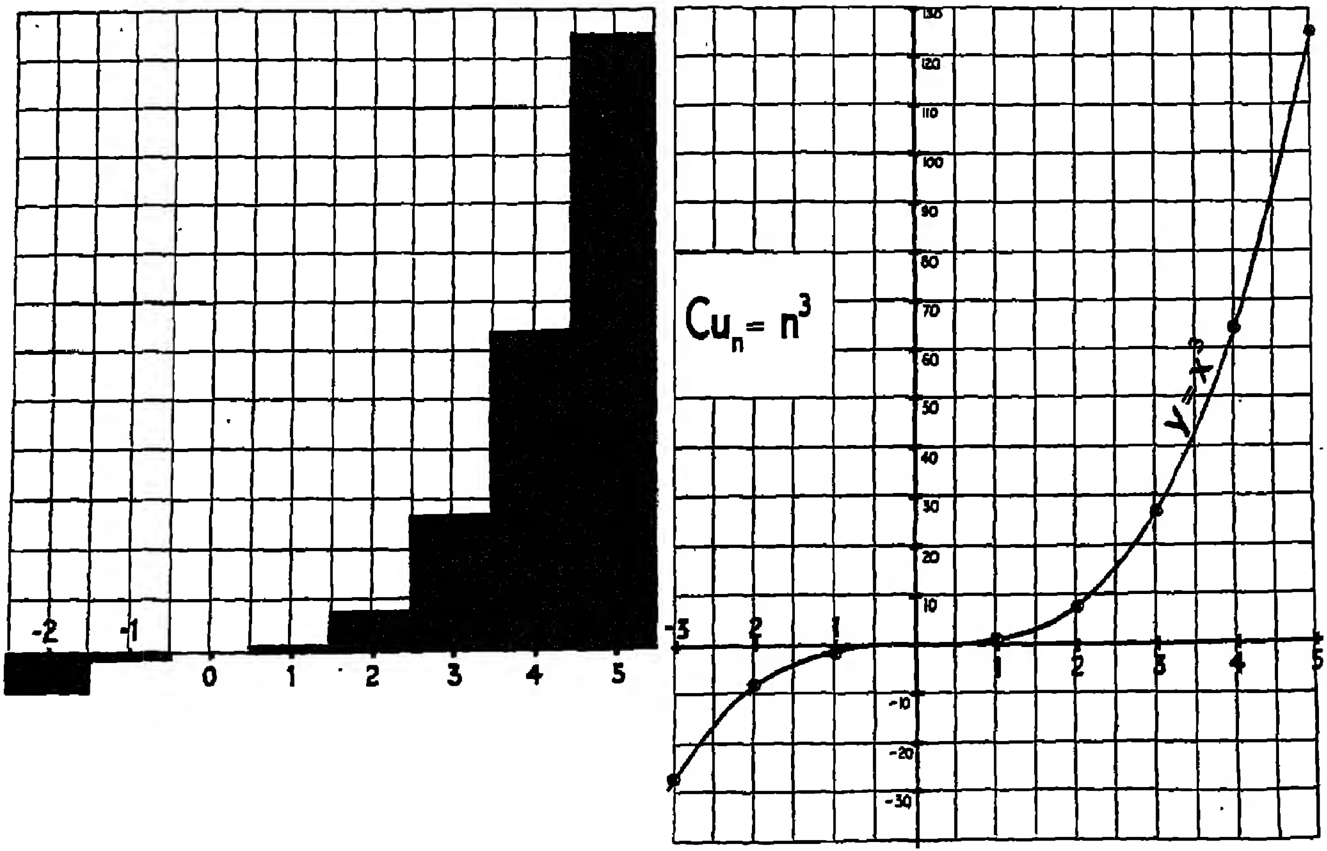
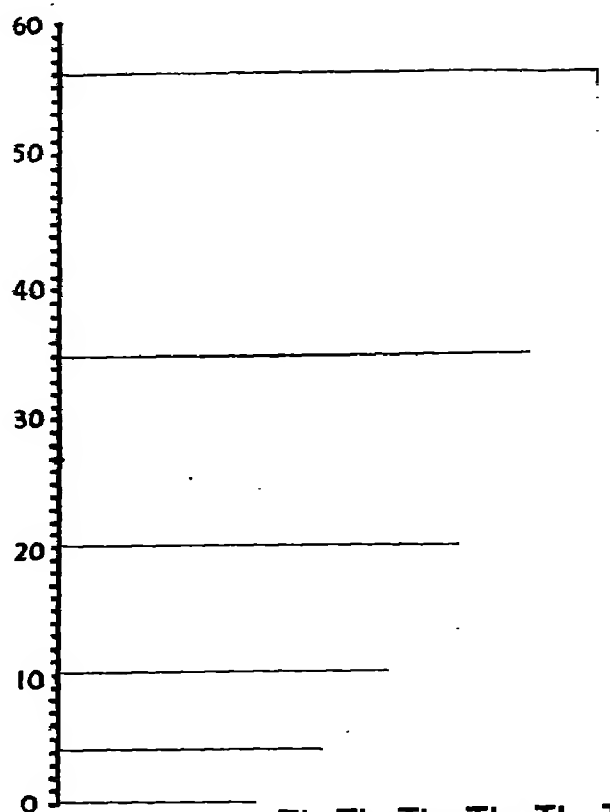


Chart 69



$Cu_n = n^3$
$Y = X^3$
-2 -1 0 1 2 3 4 5
-3 -2 -1 1 2 3 4 5
130 120 110 100 90 80 70 60 50 40 30 20 10 -10 -20 -30

Chart 70



Th_1 Th_2 Th_3 Th_4 Th_5 Th_6
1 4 10 20 35 56

$$Th_n = \frac{n(n+1)(n+2)}{6}$$

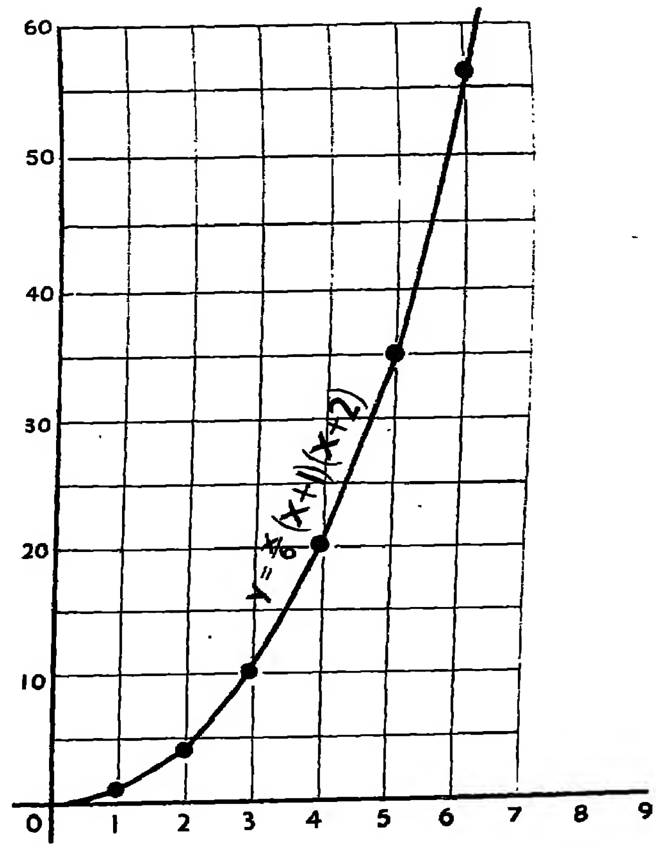


Chart 71

The Structure of the Cubic

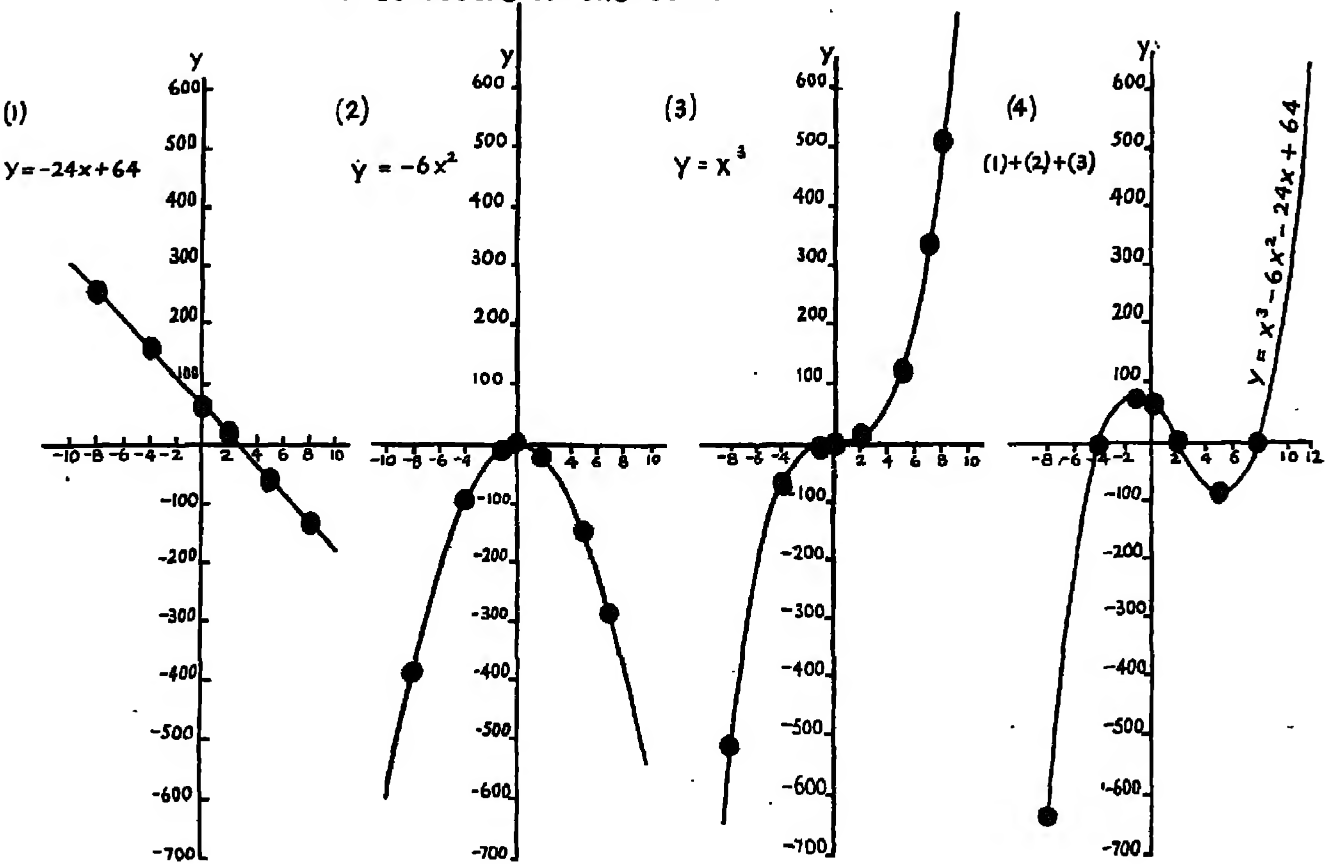


Chart 72

Areas as y-values

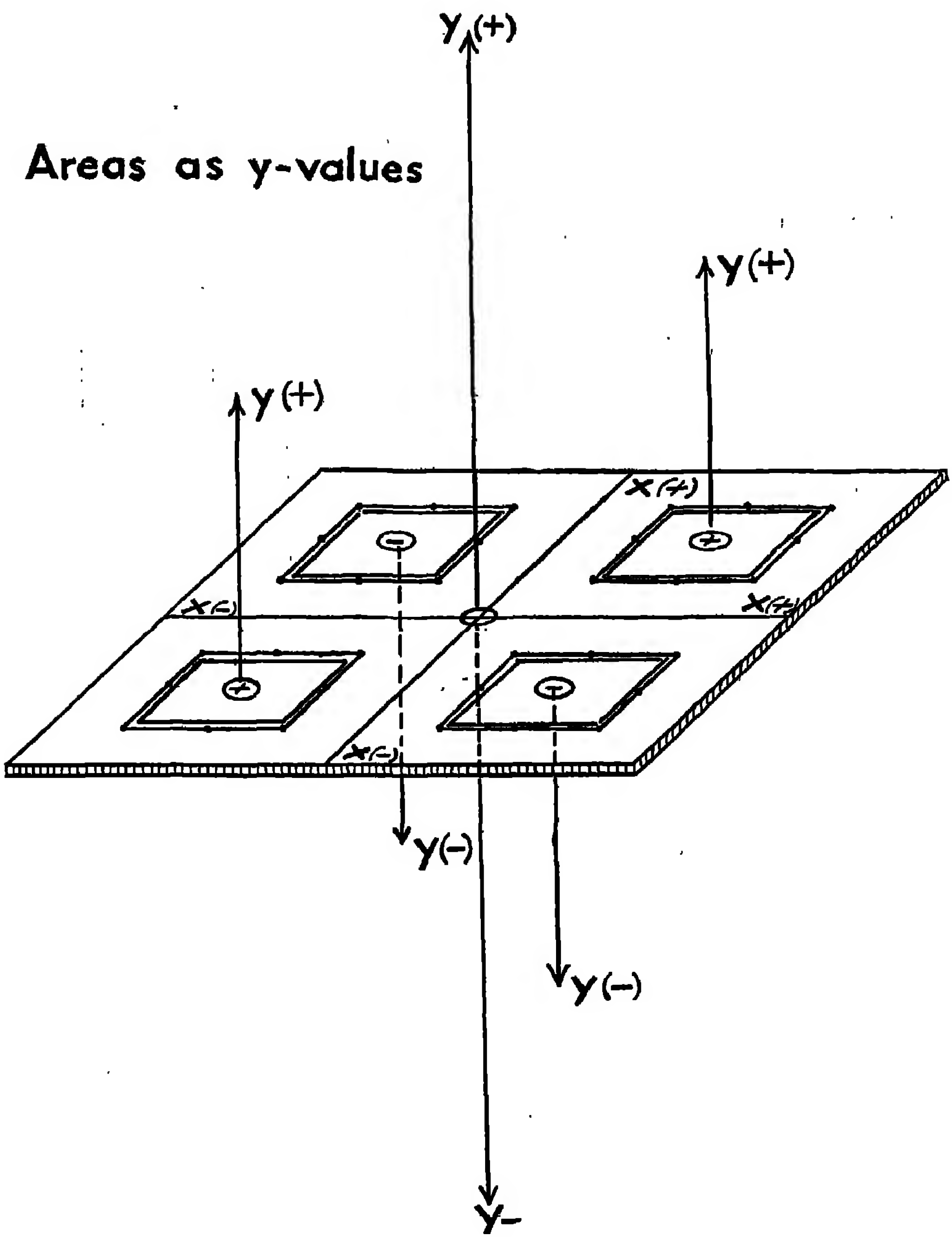
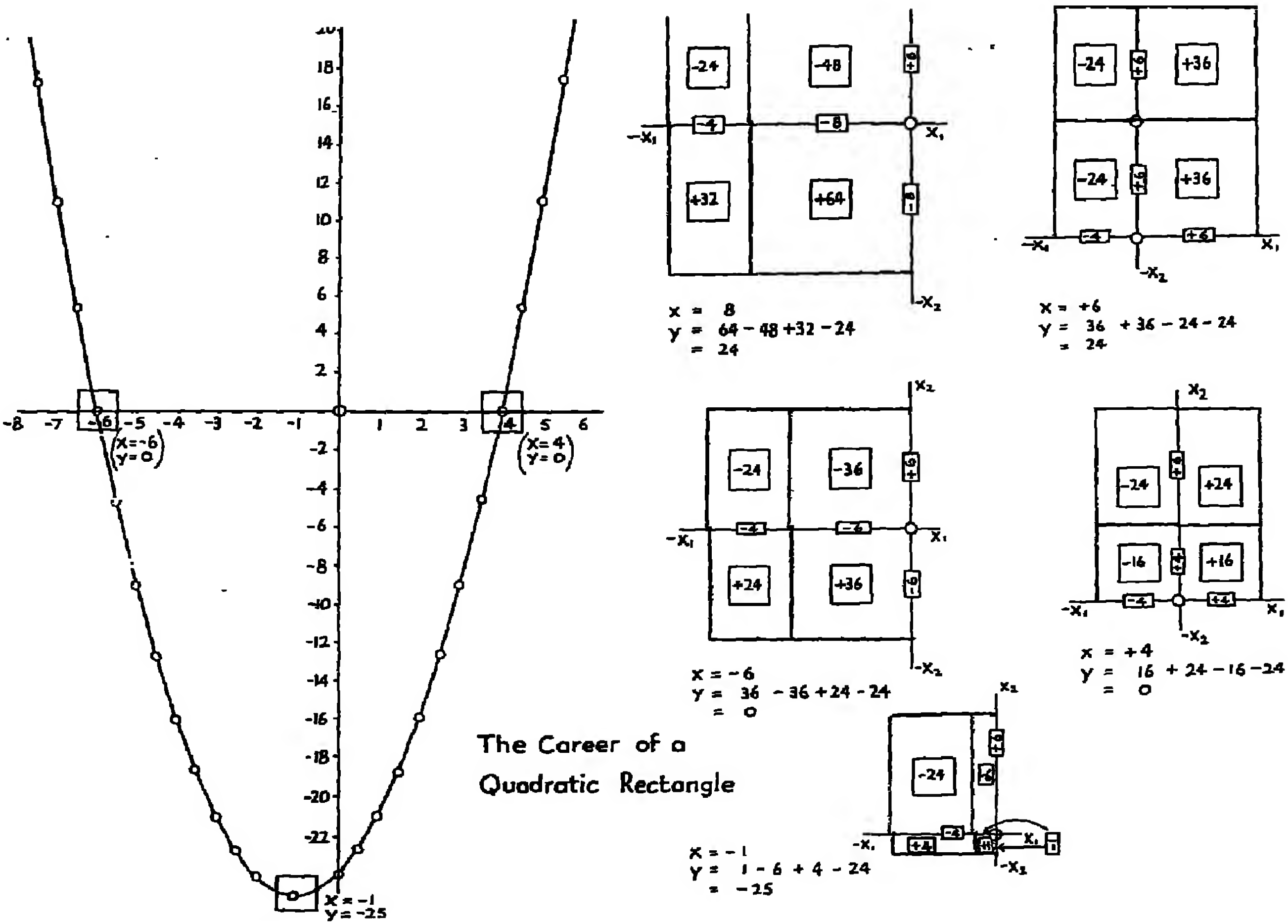


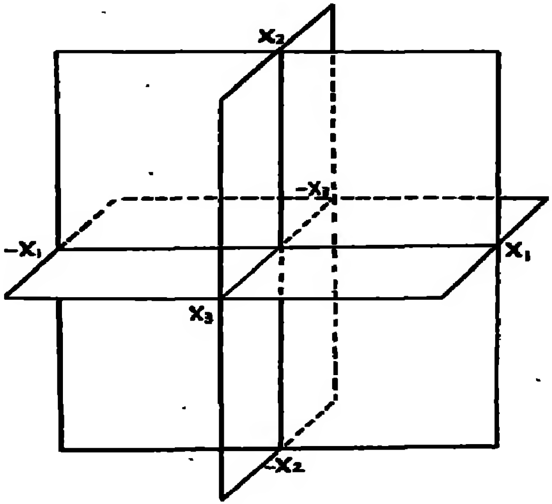
Chart 73



The Career of a Quadratic Rectangle

Chart 74

The 8 Octants and their Signs



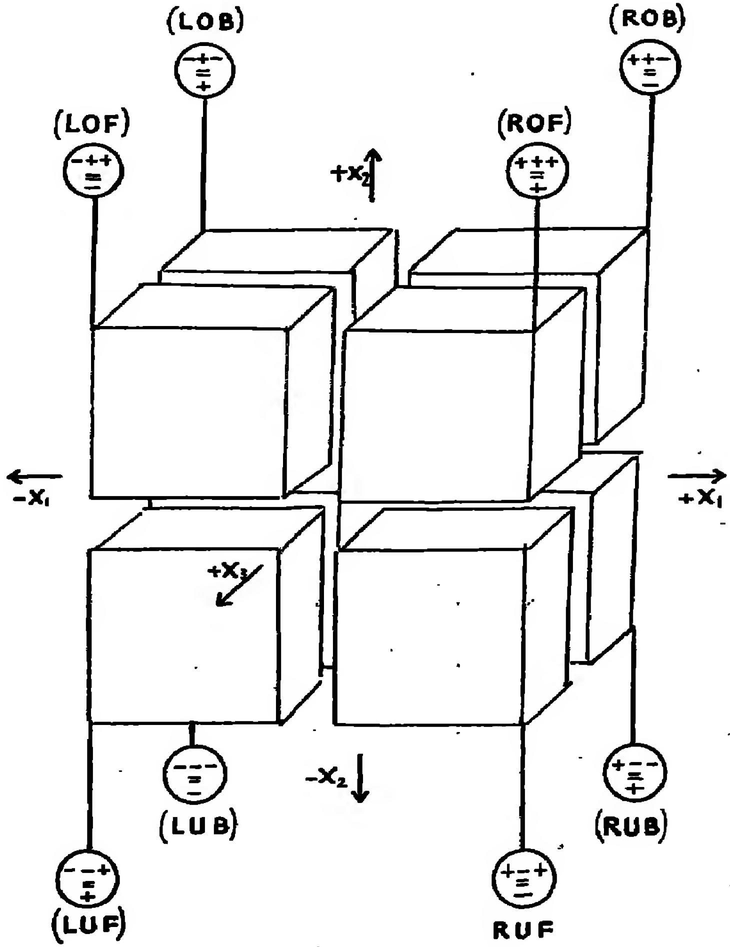
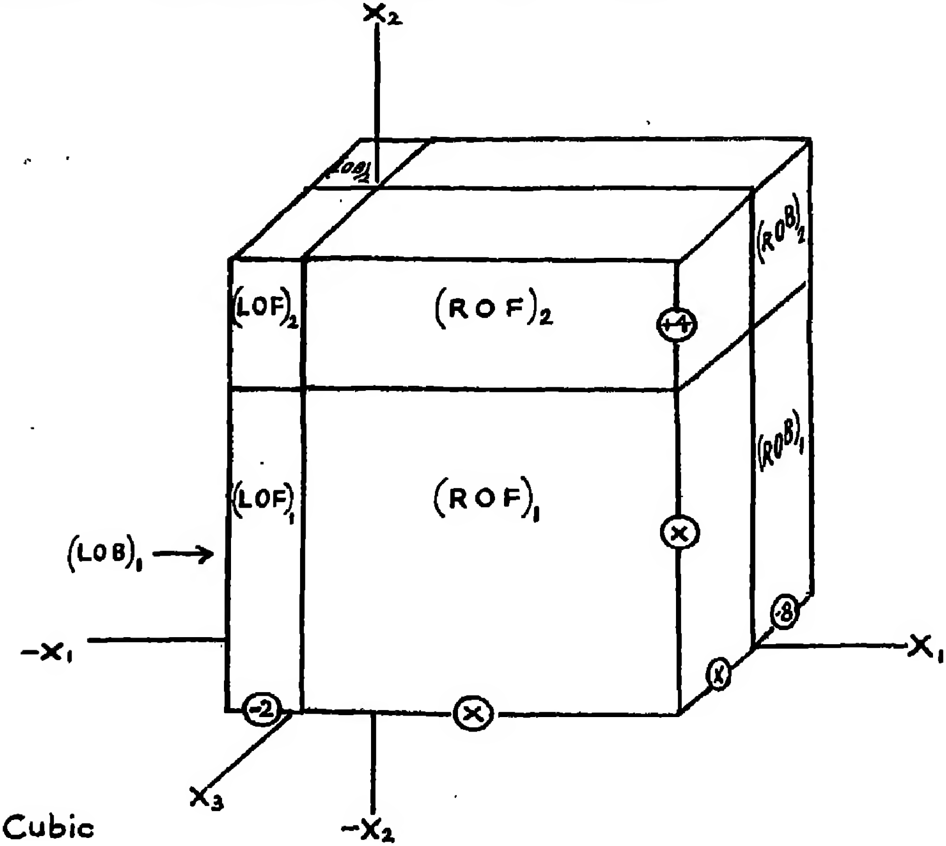


Chart 75

ɔmetric Representation of the Cubic



Cubic

$$y = (x-2)(x+4)(x-8)$$

·. the value $x = 10$

$$y = (10-2)(10+4)(10-8)$$

$$= 10^3 + 4.10^2 - 2.10^2 - 2.4.10$$

$$- 8.10^2 - 8.4.10 + 8.2.10 - 8.2.4$$

$$= 224$$

Chart 76

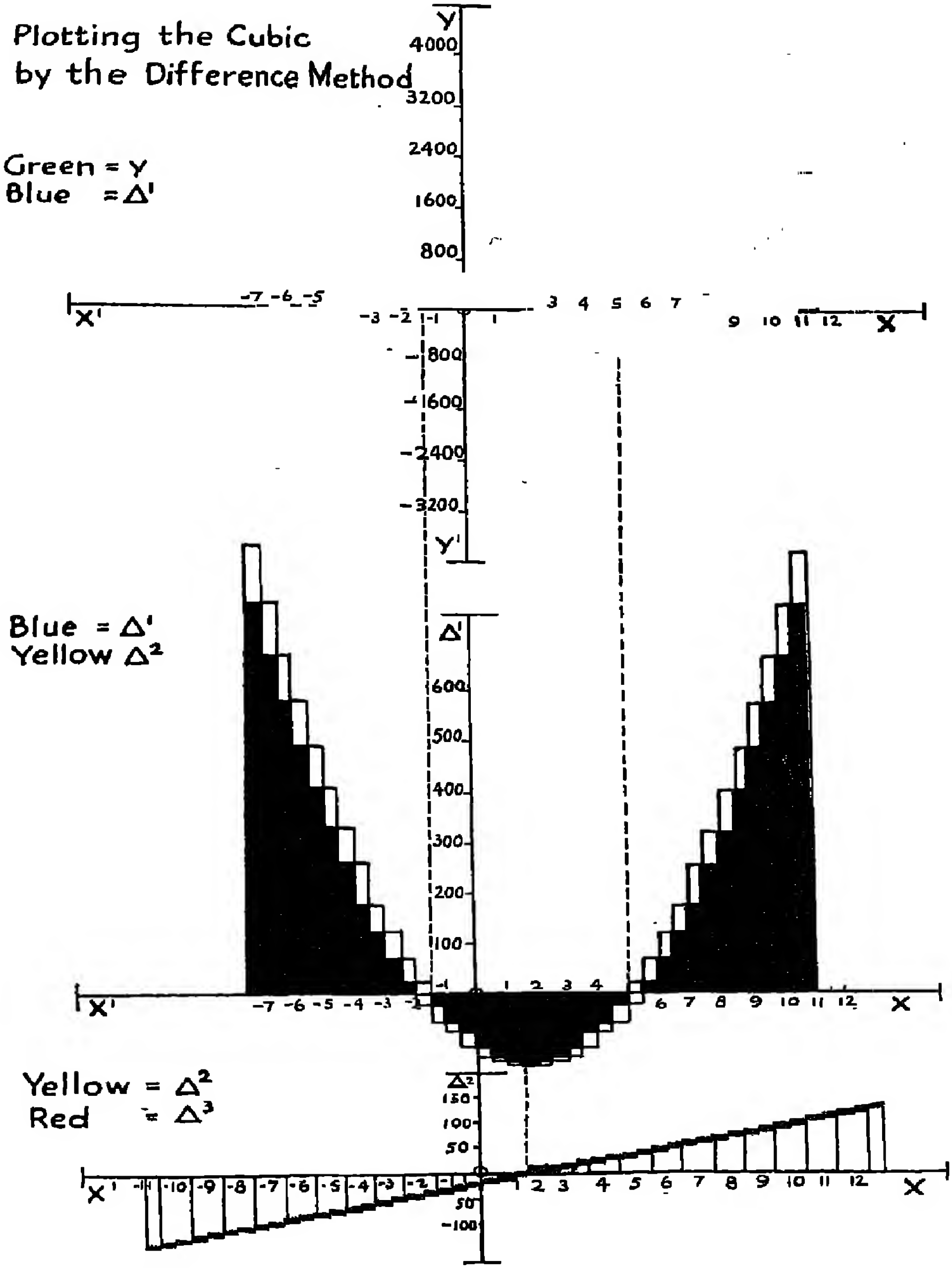

Plotting the Cubic
by the Difference Method
Green = y
Blue = Δ'
Blue = Δ'
Yellow Δ²
Yellow = Δ²
Red = Δ³

Chart 77

Solution of Cubic by Standard Method

$y = x^3 - 6x^2 - 24x + 64$

reduces to

$y = X^3 - 36X$

where $X = x - 2$

Solution

$X = -6,\ 0$ and $+6$

$\therefore x = +8,\ +2$ and -4

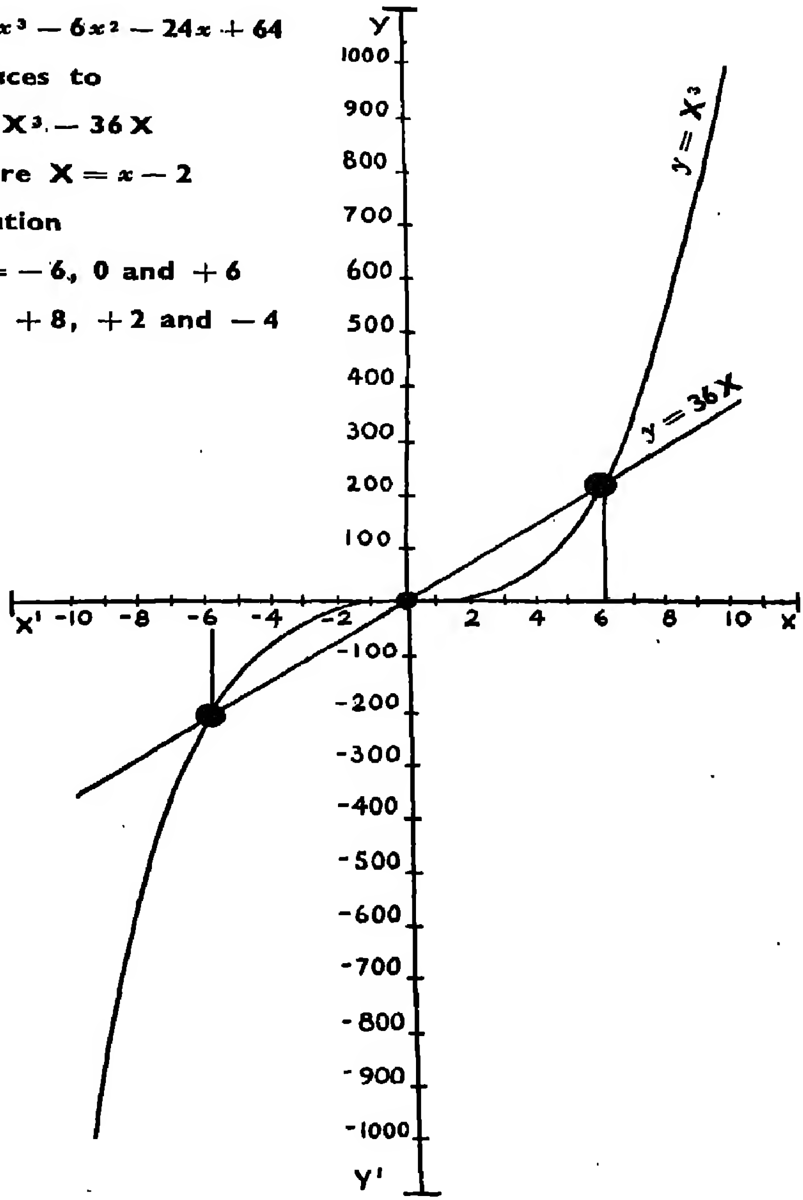
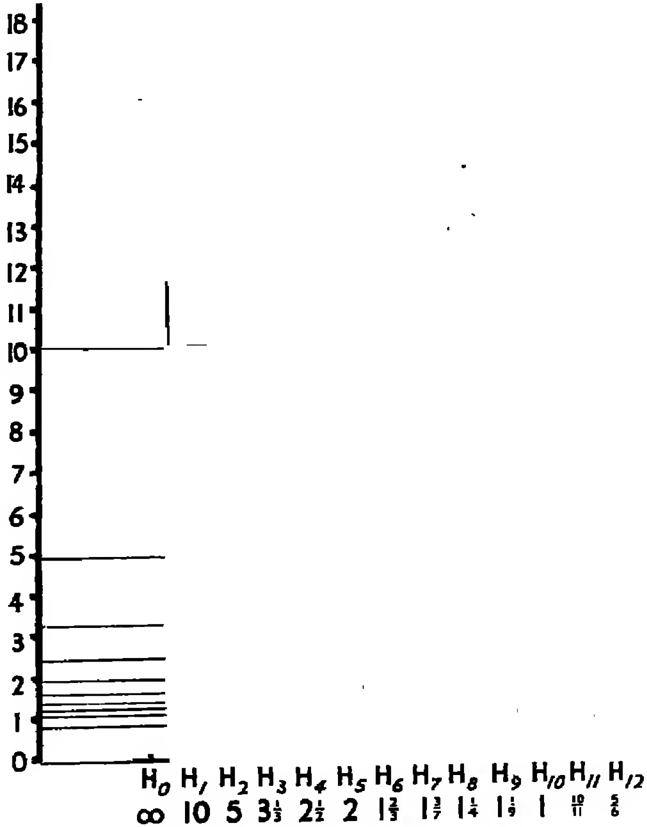
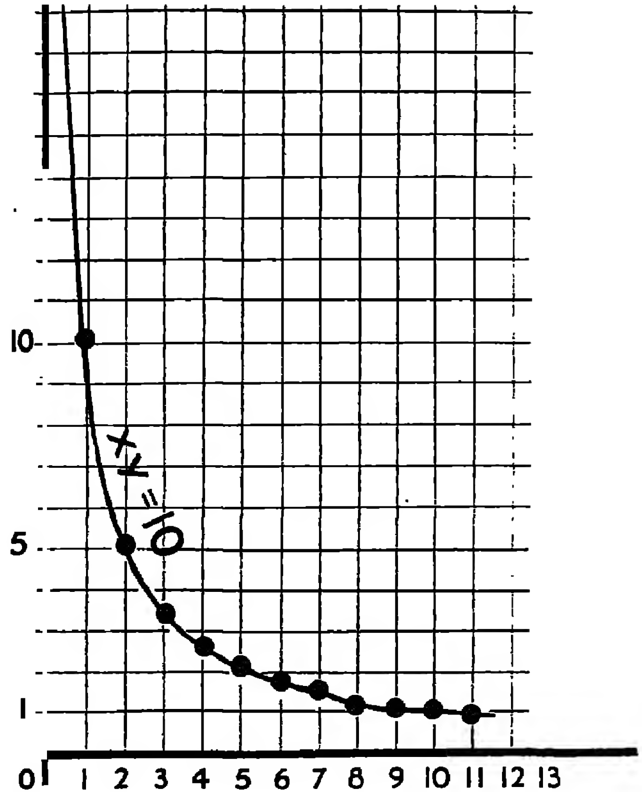


Chart 78





$$H_n = \frac{10}{n}$$

Chart 79

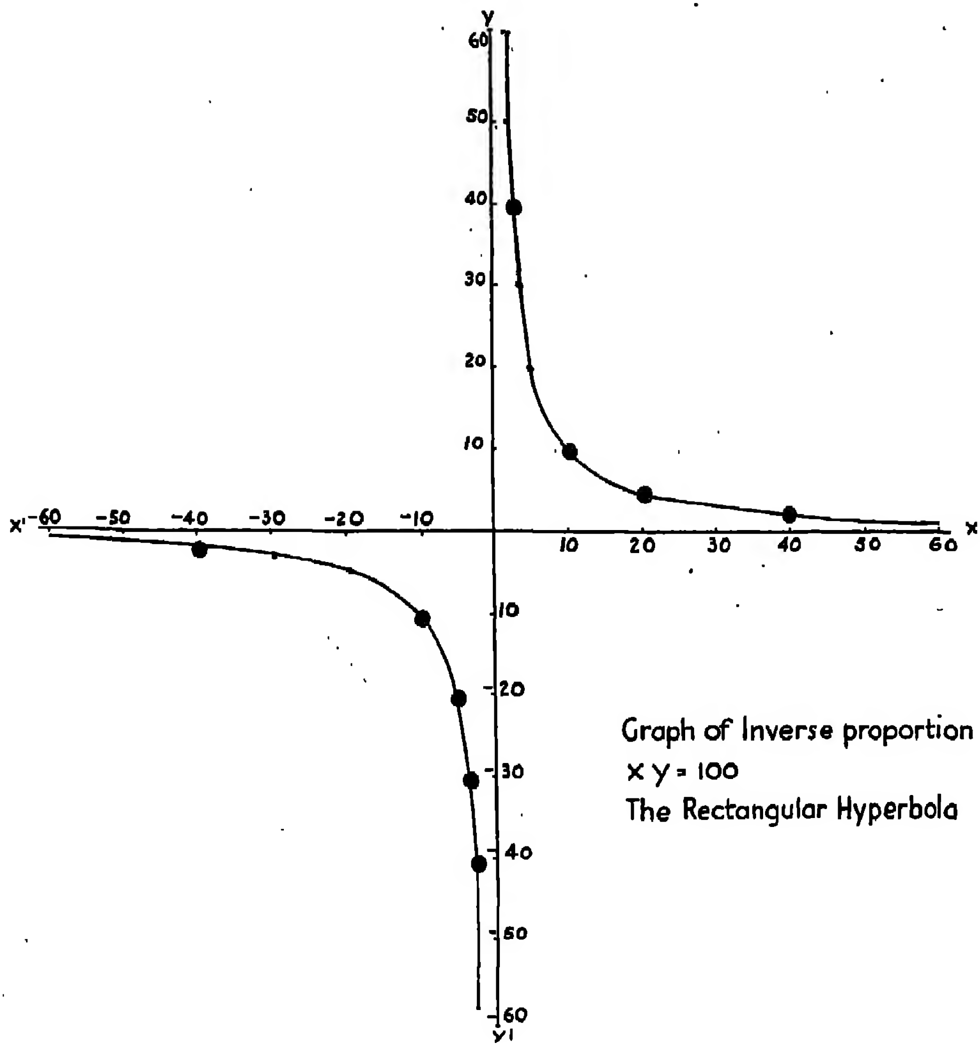

Y
60
50
40
30
20
10
X'
-60
-50
-40
-30
-20
-10
X
10
20
30
40
50
60
10
20
30
40
50
60
Y'
Graph of Inverse proportion
X Y = 100
The Rectangular Hyperbola

Chart 80

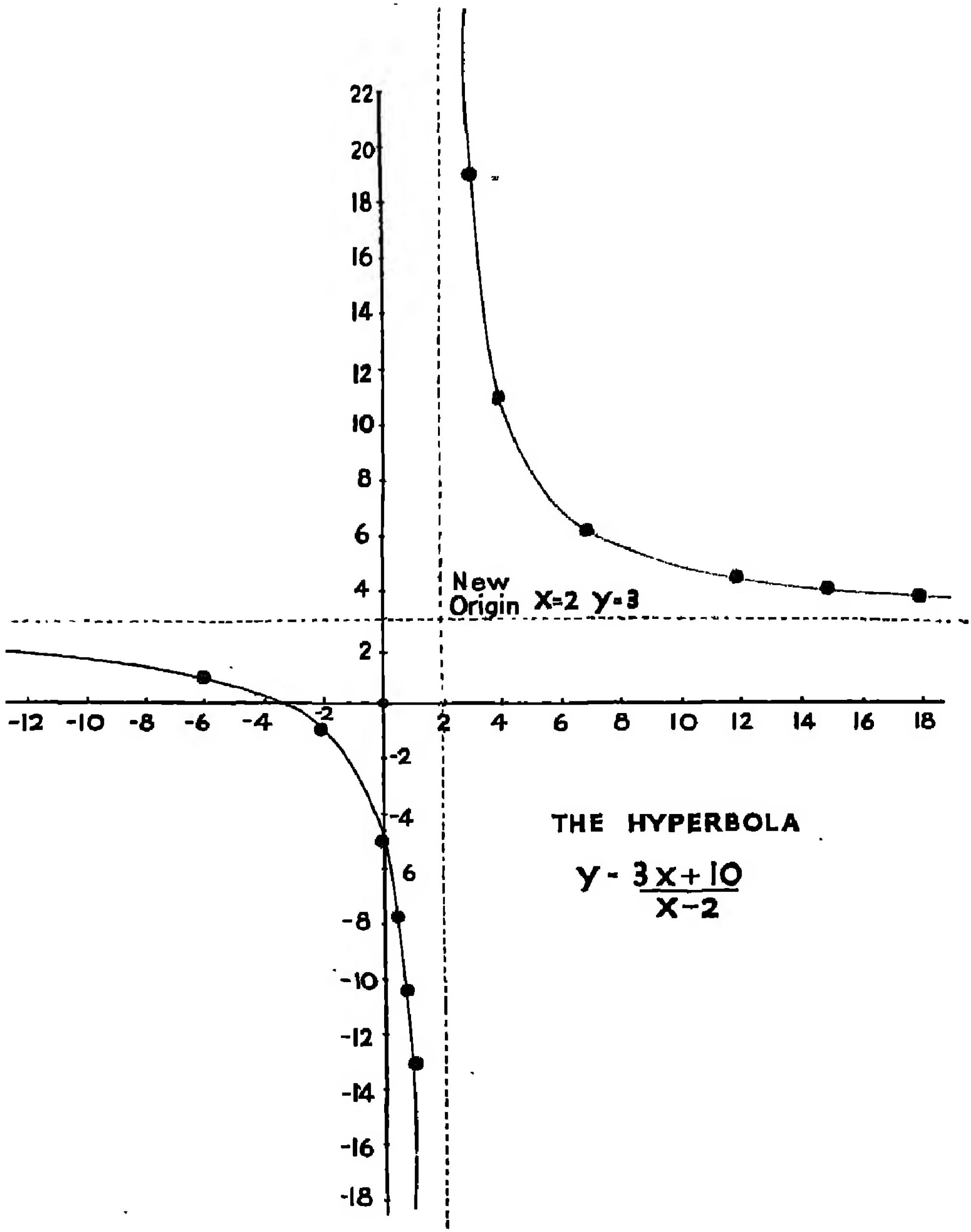
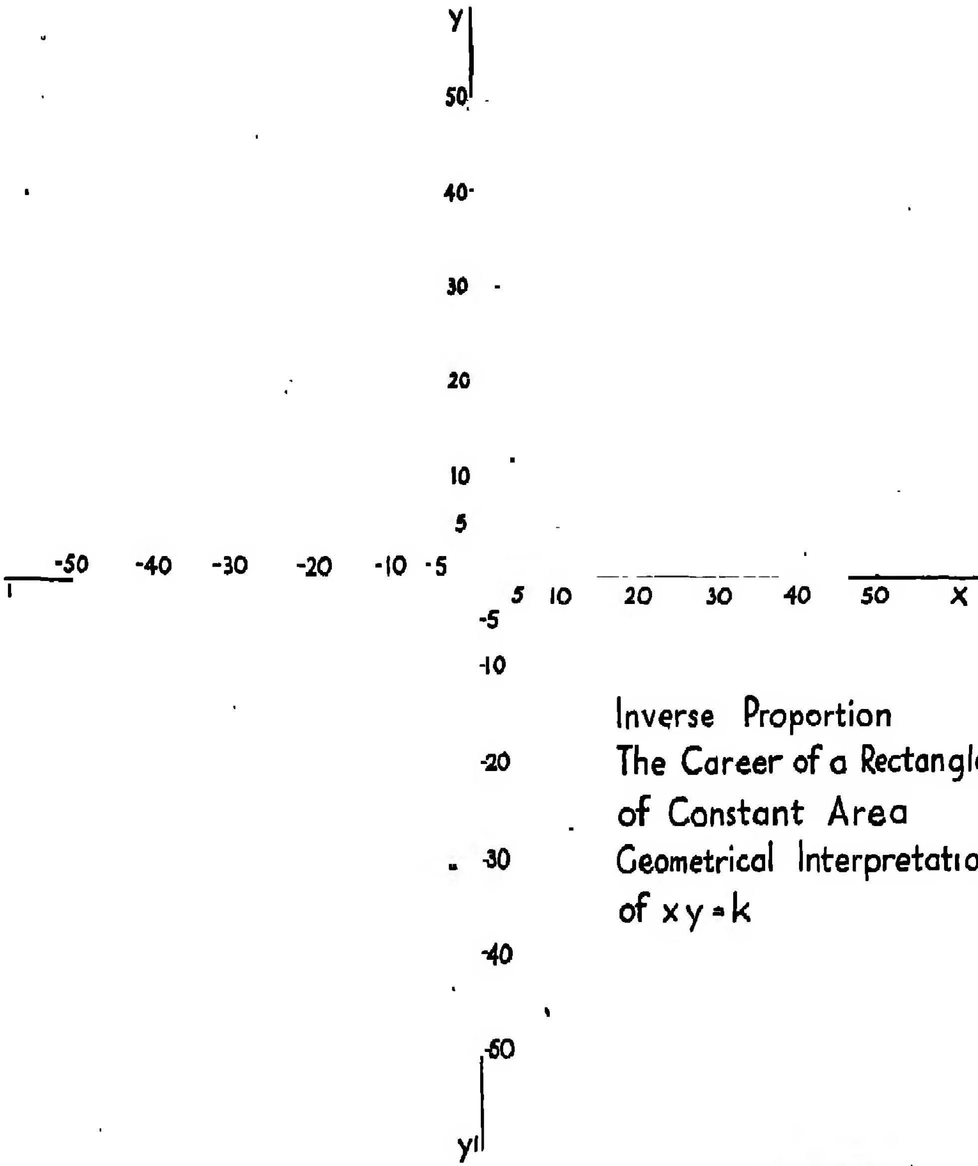

22
20
18
16
14
12
10
8
6
4
2
-2
-4
6
-8
-10
-12
-14
-16
-18
-12 -10 -8 -6 -4 -2 2 4 6 8 10 12 14 16 18
New
Origin X=2 y=3
THE HYPERBOLA
y = 3x+10 / x-2

Chart 81



Inverse Proportion
The Career of a Rectangle of Constant Area
Geometrical Interpretation of $xy = k$

Chart 82